Edge Currents for Quantum Hall Systems, II. Two-Edge, Bounded and Unbounded Geometries

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Abstract

Devices exhibiting the integer quantum Hall effect can be modeled by one-electron Schrödinger operators describing the planar motion of an electron in a perpendicular, constant magnetic field, and under the influence of an electrostatic potential. The electron motion is confined to bounded or unbounded subsets of the plane by confining potential barriers. The edges of the confining potential barriers create edge currents. This is the second of two papers in which we review recent progress and prove explicit lower bounds on the edge currents associated with one- and two-edge geometries. In this paper, we study various unbounded and bounded, two-edge geometries with soft and hard confining potentials. These two-edge geometries describe the electron confined to unbounded regions in the plane, such as a strip, or to bounded regions, such as a finite length cylinder. We prove that the edge currents are stable under various perturbations, provided they are suitably small relative to the magnetic field strength, including perturbations by random potentials. The existence of, and the estimates on, the edge currents are independent of the spectral type of the operator.

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Contents

1 Introduction and Main Results

This is the second of two papers dealing with lower bound estimates on edge currents associated with quantum Hall devices. The integer quantum Hall effect (IQHE) refers to the quantization of the Hall conductivity in integer multiples of $2\pi e^2/h$. The IQHE is observed in planar quantum devices at zero temperature and can be described by a Fermi gas of noninteracting electrons. This simplification reduces the study of the dynamics to the one-electron approximation. Typically, experimental devices consist of finitely-extended, planar samples subject to a constant perpendicular magnetic field B. An applied electric field in the x-direction induces a current in the y-direction, the Hall current, and the Hall conductivity σ_{xy} is observed to be quantized. Furthermore, the Hall conductivity is a function of the electron Fermi energy, or, equivalently, the electron filling factor, and plateaus of the Hall conductivity are observed as the filling factor is increased. It is now accepted that the occurrence of the plateaus is due to the existence of localized states near the Landau levels that are created by the random distribution of impurities in the sample. We refer to [8] and references mentioned there for a more detailed discussion. Since the earliest theoretical discussions, the existence of edge currents has played a major role in the explanation of the quantum Hall effect.

To describe the two-edge geometries dealt with in the paper, we first recall the theory for the plane. The Landau Hamiltonian $H_L(B)$ describes a particle constrained to $\mathbb{R}^2$, and moving in a constant, transverse magnetic field with strength $B \geq 0$. Let $p_x = -i\partial_x$ and $p_y = -i\partial_y$ be the two momentum operators. The operator $H_L(B)$ is defined on the dense domain $C_0^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ by

$$H_L(B) = (-i\nabla - A)^2 = p_x^2 + (p_y - Bx)^2, \tag{1.1}$$

in the Landau gauge for which the vector potential is $A(x, y) = B(0, x)$. This extends to a selfadjoint operator with point spectrum given by $\{E_n(B) = (2n + 1)B \mid n = 0, 1, 2, \ldots\}$, and each eigenvalue is infinitely degenerate.

As in [8], we define the *edge current* as the expectation of the y-component of the velocity operator $V_y \equiv (p_y - Bx)$ in certain states that will be specified below. These are states with energy concentration between two successive Landau levels $E_n(B)$ and $E_{n+1}(B)$.

1.1 Main Results

Our main results in this paper can be grouped together as follows.

1. Two-Edge, Unbounded Geometries: We study the strip case for which the electron is constrained to the region $-L/2 < x < L/2$, a strip of width $L > 0$. The characteristic function of the set J being denoted by χ_J, ¿the confining potential has either one of the two forms:

   (a) Sharp Confining Potential

   $$V_0(x) = \mathcal{V}_0 \chi_{\{|x|>L/2\}}(x),\ \mathcal{V}_0 > 0, \tag{1.2}$$

   (b) Power Function Confining Potentials

   $$V_0(x) = \mathcal{V}_0 (|x| - L/2)^p \chi_{\{|x|>L/2\}}(x),\ \mathcal{V}_0 > 0,\ p > 1. \tag{1.3}$$

2. Two-Edge, Bounded Geometries: We study models for which the electron on a cylinder $C_D = \mathbb{R} \times DS^1$, for $D > 0$, is confined to the bounded region $[-L/2, L/2] \times DS^1$ by the sharp confining potential (1.2).

As a preamble to the investigation of these models, we shall systematically examine the straight parabolic channel model studied by Exner, Joye and Kovarik in [2]. In this case the confining potential is defined by

$$V_0(x) = g^2 x^2,\ g > 0, \tag{1.4}$$

and it turns out this model is completely solvable, making the estimation of the edge currents rather straightforward in this particular case.

As in [8], we first study the edge currents for the unperturbed Hamiltonian $H_0 = H_L(B) + V_0$. We then examine the stability of the lower bounds under potential perturbations. In the sharp potential case, we prove that the lower bounds are uniform with respect to the confining potential. This means that we can take the limit as the size of the confining potential becomes infinite.

As a result, our results extend to the case of Dirichlet boundary conditions along the edges. The proof of this follows as in the first paper [8].

In all cases, the unperturbed Hamiltonian has the form

$$H_0 = H_L(B) + V_0, \tag{1.5}$$

acting on the Hilbert space $L^2(\mathbb{R}^2)$. This is a nonnegative, selfadjoint operator. Our strategy is to analyze the unperturbed operator via the partial Fourier transform in the y-variable. We write $\hat{f}(x,k)$ for this partial Fourier transform. For the case of unbounded geometry, we have $k \in \mathbb{R}$, whereas for the case of bounded geometry, the allowable k values are discrete. In either case, this decomposition reduces the problem to a study of the fibered operators of the form

$$h_0(k) = p_x^2 + (k - Bx)^2 + V_0(x), \tag{1.6}$$

acting on $L^2(\mathbb{R})$. Since the effective, nonnegative, potential $V_{eff}(x;k) \equiv (k-Bx)^2+V_0(x)$ is unbounded as $x \to \pm\infty$, the resolvent of $h_0(k)$ is compact and the spectrum is discrete. We denote the eigenvalues of $h_0(k)$ by $\omega_j(k)$, with corresponding normalized eigenfunctions $\varphi_j(x;k)$, so that

$$h_0(k)\varphi_j(x;k) = \omega_j(k)\varphi_j(x;k), \quad \|\varphi_j(\cdot;k)\| = 1. \tag{1.7}$$

As in [8], the properties of the curves $k \in \mathbb{R} \to \omega_j(k)$ play an important role in the proofs. These curves are called the *dispersion curves* for the unperturbed Hamiltonian (1.5). The importance of the properties of the dispersion curves comes from an application of the Feynman-Hellmann formula. To illustrate this, let us first consider the two-edge geometry of a half-plane with the sharp confining potential. We note that unlike for the case of one-edge geometries, the dispersion curves are no longer monotonic in k. For simplicity, we consider in this introduction a closed interval $\Delta_0 \subset (B, 3B)$ and a normalized wave function ψ satisfying $\psi = E_0(\Delta_0)\psi$, where $E_0(\Delta_0)$ denotes the spectral projection of H_0 associated with Δ_0. Such a function admits a decomposition of the form

$$\psi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{\omega_0^{-1}(\Delta_0)} e^{iky} \beta_0(k) \varphi_0(x;k)\; dk, \tag{1.8}$$

where the coefficient $\beta_0(k)$ is defined by

$$\beta_0(k) \equiv \langle \hat{\psi}(\cdot;k), \varphi_0(\cdot;k) \rangle. \tag{1.9}$$

The matrix element of the current operator V_y in such a state is

$$\langle \psi, V_y \psi \rangle = \int_{\mathbb{R}} dx \int_{\omega_0^{-1}(\Delta_0)} dk |\beta_0(k)|^2 (k - Bx) |\varphi_0(x;k)|^2. \tag{1.10}$$

From (1.7) and the Feynman-Hellmann Theorem, we find that

$$\omega_0'(k) = 2 \int_{\mathbb{R}} dx \ (k - Bx) \ |\varphi_0(x;k)|^2, \tag{1.11}$$

so that we get

$$\langle \psi, V_y \psi \rangle = \frac{1}{2} \int_{\mathbb{R}} |\beta_0(k)|^2 \ \omega_0'(k) \ dk. \tag{1.12}$$

It follows from (1.12) that in order to obtain a lower bound on the expectation of the current operator in the state ψ we need to bound the derivative $\omega_0'(k)$ from below for $k \in \omega_0^{-1}(\Delta_0)$. The next step of the proof involves relating the derivative $\omega_0'(k)$ to the trace of the eigenfunction $\varphi_0(x;k)$ on the boundary of the strip. For this, we use the formal commutator expression

$$\hat{V}_y(k) \equiv (k - Bx) = \frac{-i}{2B}[p_x, h_0(k)] + \frac{1}{2B} V_0'(x). \tag{1.13}$$

Inserting this into the identity (1.11), we find

$$\begin{aligned}
\omega_0'(k) &= 2\langle \varphi_0(\cdot;k), (k - Bx)\varphi_0(\cdot;k)\rangle \\
&= \frac{-i}{2B}\langle \varphi_0(\cdot;k), [p_x, h_0(k) - \omega_0(k)]\varphi_0(\cdot;k)\rangle + \frac{1}{B}\langle \varphi_0(\cdot;k), V_0'\varphi_0(\cdot;k)\rangle \\
&= \frac{\mathcal{V}_0}{B}(\varphi_0(L/2;k)^2 - \varphi_0(-L/2,k)^2),
\end{aligned} \tag{1.14}$$

since the commutator term on the second line vanishes by the Virial Theorem. Upon inserting (1.14) into the expression (1.12) for the edge current, we obtain

$$\langle \psi, V_y \psi \rangle = \frac{\mathcal{V}_0}{2B} \int_{\omega_0^{-1}(\Delta_0)} |\beta_0(k)|^2 (\varphi_0(L/2;k)^2 - \varphi_0(-L/2,k)^2) \ dk. \tag{1.15}$$

Consequently, we are left with the task of estimating the trace of the eigenfunction along the two boundary components at $x = \pm L/2$.

The key point that allows us to distinguish these two traces is the following. The dispersion curves are symmetric about $k = 0$ if $V_0(x)$ is an even

function. Consequently, if a wave function ψ satisfies $\psi = E_0(\Delta_0)\psi$, we have to study the decomposition of ψ in k-space according to the decomposition $\omega_0^{-1}(\Delta_0) = \omega_0^{-1}(\Delta_0)_- \cup \omega_0^{-1}(\Delta_0)_+$, where $\omega_0^{-1}(\Delta_0)_\pm \equiv \omega_0^{-1}(\Delta_0) \cap \mathbb{R}_\pm$. These two components correspond to currents propagating in opposite directions along the left and right edges of the band, respectively. To construct a left-edge current, we construct states ψ so that the coefficients $\beta_0(k)$ in (1.9) satisfy supp $\beta_0(k) \subset \omega_0^{-1}(\Delta_0)_-$. Such a state is spatially concentrated near the left edge $x = -L/2$. Hence, the contribution to the left-edge current coming from $\varphi_0(L/2; k)$ will be exponentially small since the domain $x \approx L/2$ is in the classically forbidden region for energies $\omega_0(k)$, for $k \in \omega_0^{-1}(\Delta_0)_-$. Consequently, the contribution to the integral in (1.15) will be exponentially small. Thus, we prove that if $\psi = E_0(\Delta_0)\psi$ is spectrally concentrated in the set $\omega_0^{-1}(\Delta_0)_-$, then the matrix element $\langle \psi, V_y \psi \rangle$ is bounded from below by a constant times $\sqrt{B}\|\psi\|^2$. Much of our technical work, therefore, is devoted to obtaining lower bounds on quantities of the form $\mathcal{V}_0 \varphi_0(\pm L/2; k)^2$ for such left-edge current states. We also mention that similar results hold for the right-edge current. Of course, in the unperturbed case with a symmetric confining potential, we expect that the net current across any line $y = C$ is zero for the unperturbed problem. We will prove this in Proposition 2.1 below.

1.2 Contents

This paper is organized as follows. Section 2 is devoted to the estimation of edge currents for the case of the sharp confining potential (1.2), the power function confining potential (1.3), and the parabolic confining potential (1.4). In section 3, the spectral properties of these models are investigated. Using the Mourre commutator method, we exhibit a class of potentials V_1 (periodic or decreasing in the y-direction) preserving nonempty absolutely continuous spectrum in intervals lying between two consecutive Landau levels for the perturbed Hamiltonian $H_0 + V_1$. In section 4, we address cylinder geometries models and prove the existence of edge currents for Hamiltonians with pure point spectrum in this framework. Appendix 1 in section 5 presents basic properties of the dispersion curves needed in the proofs. In Appendix 2, section 6, we collect technical results needed in section 2 for the estimation of edge currents for the power function confining potential.

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2 Edge Currents for Two-Edge Geometries

Many quantum devices can be modeled by a confining potential forcing the electrons into a strip of infinite extent in one direction. The dynamics of electrons in an infinite-strip are different from the half-plane cases treated in [8]. We study an electron in a strip of width $L > 0$ in the x-direction, and unbounded in the y-direction. We consider confining potential $V_0(x)$ that are either step functions, or power functions. After some basic analysis of these models that is independent of the precise form of the confining potential, we study edge currents for parabolic confining potential, sharp confining potential and power function confining potential.

2.1 Basic Analysis of Two-Edge Geometries

As in [8], we study the existence of edge currents for a general confining potential $V_0(x)$. We obtain lower bounds on the appropriately localized velocity along the y-direction $V_y = p_y - Bx$. The strip geometry is a two-edge geometry. Thus, we expect that there is a current associated with each edge. Classically, these currents propagate along the edges in opposite directions. For the unperturbed system, one expects that the net current flow across the line $y = C$, for any $C \in \mathbb{R}$, to be zero, and we prove this in Proposition 2.1. Once a perturbation V_1 is added, this may no longer be true, and the persistence of edge currents may depend upon a relationship between B and L.

We continue to use the same notation as in [8]. That is, we write $H_0 = H_L(B) + V_0$ for the unperturbed operator. Since we have translational invariance in the y-direction, this operator admits a direct sum decomposi-

tion

$$H_0 = \int_{\mathbb{R}}^{\oplus} dk h_0(k). \tag{2.1}$$

We write $h_0(k)$ for the fibered operator acting on $L^2(\mathbb{R})$, where

$$h_0(k) = p_x^2 + (k - Bx)^2 + V_0(x), \tag{2.2}$$

with an even, two-edge confining potential V_0. Although some of our arguments hold for a general confining potential that is monotone on the left and the right, we will explicitly treat two cases, the sharp confining potential given in (1.2), and the power function confining potential given in (1.3). We first prove that the total edge current carried by certain symmetric states of finite energy vanishes. For this, it is essential that the confining potential be an *even* function. We consider states of finite energy ψ, with $\psi \in E_0(\Delta_n) L^2(\mathbb{R}^2)$, for an interval $\Delta_n \subset (E_n(B), E_{n+1}(B))$, for any $n \geq 0$. The partial Fourier transform $\hat{\psi}$ of ψ in the y-variable can be expressed in terms of the eigenfunctions $\varphi_j(x;k)$ as

$$\hat{\psi}(x,k) = \sum_{j=0}^{n} \chi_{\omega_j^{-1}(\Delta_n)}(k) \beta_j(k) \varphi_j(x;k), \tag{2.3}$$

or equivalently as

$$\psi(x,y) = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n} \int_{\mathbb{R}} e^{iky} \; \chi_{\omega_j^{-1}(\Delta_n)}(k) \beta_j(k) \varphi_j(x;k) \; dk, \tag{2.4}$$

where the coefficients $\beta_j(k)$ are defined by

$$\beta_j(k) \equiv \langle \hat{\psi}(\cdot;k), \varphi_j(\cdot;k) \rangle. \tag{2.5}$$

and the normalization condition

$$\|\psi\|^2_{L^2(\mathbb{R}^2)} = \sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)} |\beta_j(k)|^2 \; dk. \tag{2.6}$$

We recall that the properties of the dispersion curves $\omega_j(k)$ result in the disjoint decomposition $\omega_j^{-1}(\Delta_n) = \omega_j^{-1}(\Delta_n)_- \cup \omega_j^{-1}(\Delta_n)_+$ with $\omega_j^{-1}(\Delta_n)_\pm \equiv \omega_j^{-1}(\Delta_n) \cap \mathbb{R}_\pm$. It is clear from the fact the potential in $h_0(k)$ is centered at $x_0 = k/B$ that the wave function ψ may be more localized near one edge or

another depending upon the properties of the weights $\beta_j(k)$. For example, if the $\beta_j(k)$ are supported only by negative wave numbers k, then the wave function will be localized near the left edge. Such a wave function should carry a net left-edge current. We will prove this below. We will first prove that if a wave function is symmetrically localized with respect to the left and right edges, then it carries no net edge current: The left-edge current cancels the right-edge current.

Let us make the assumption on the confining potential $V_0(x)$ more precise. In the sequel we assume V_0 is an even function which satisfies simultaneously the two following conditions:

$$\begin{cases} (a) & 0 \leq V_0(x) \leq C, \ \forall x \in \mathbb{R} \\ (b) & \lim_{|x| \to \infty} V_0(x) = C, \end{cases} \tag{2.7}$$

for some generalized constant $0 < C \leq \infty$.

It is clear that the potential V_0 is unbounded at infinity in the case where $C = \infty$, while it is uniformly bounded by C otherwise. Actually, each of the particular confining potentials we will consider below satisfy (2.7). Indeed, this is the case for the sharp confining potential (1.2) for $C = \mathcal{V}_0$, as well as for the power function confining potential (1.3) and the parabolic confining potential (1.4) by taking $C = \infty$.

Proposition 2.1 *Let $V_0(x)$ be a even confining potential satisfying (2.7). Let $\omega_j(k)$, for $j = 0, 1, 2, \ldots$, be the dispersion curves for $h_0(k)$. Let $\psi \in E_0(\Delta_n) L^2(\mathbb{R}^2)$, as in (2.3), be a finite energy state. Then, the current carried by such a state has the following expression:*

$$\langle \psi, V_y \psi \rangle = \frac{1}{2} \sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)_-} (|\beta_j(k)|^2 - |\beta_j(-k)|^2) \omega_j'(k) dk. \tag{2.8}$$

Henceforth, if ψ is a symmetric state, that is, $\beta_j(k) = \beta_j(-k)$, for $j = 0, 1, \cdots, n$, then the current carried by ψ vanishes:

$$\langle \psi, V_y \psi \rangle = 0. \tag{2.9}$$

Proof.

The velocity $V_y = p_y - Bx$ has a Fourier transform that we write as $\hat{V}_y =$

$\hat{V}_y(k) = k - Bx$. Using the Fourier decomposition (2.3), the matrix element of the velocity operator V_y is

$$\begin{aligned} & \langle \psi, V_y \psi \rangle \qquad (2.10) \\ = & \sum_{j,l=0}^{n} \int_{\mathbb{R}} \chi_{\omega_j^{-1}(\Delta_n)}(k) \chi_{\omega_l^{-1}(\Delta_n)}(k) \overline{\beta}_j(k) \beta_l(k) \langle \varphi_j(\cdot;k), \hat{V}_y(k) \varphi_l(\cdot;k) \rangle \, dk. \end{aligned}$$

As a consequence of the result of Lemma 2.1 below, the cross-terms in (2.10) vanish, at least for $(|\Delta_n|/B)$ sufficiently small, giving

$$\begin{aligned} \langle \psi, V_y \psi \rangle &= \sum_{j=0}^{n} \int_{\mathbb{R}} \chi_{\omega_j^{-1}(\Delta_n)}(k) |\beta_j(k)|^2 \langle \varphi_j(\cdot;k), \hat{V}_y(k) \varphi_j(\cdot;k) \rangle \, dk \\ &= \sum_{j=0}^{n} \int_{-\infty}^{0} \chi_{\omega_j^{-1}(\Delta_n)}(k) \left\{ |\beta_j(k)|^2 \langle \varphi_j(\cdot;k), \hat{V}_y(k) \varphi_j(\cdot;k) \rangle \right. \\ & \quad \left. + |\beta_j(-k)|^2 \langle \varphi_j(\cdot;-k), \hat{V}_y(-k) \varphi_j(\cdot;-k) \rangle \right\} dk, \qquad (2.11) \end{aligned}$$

where we used the fact, proved in Lemma 5.1 in Appendix 1, that the dispersion curves are even functions of k, that is, $\omega_j(k) = \omega_j(-k)$. We also note that the Hamiltonian $h_0(k)$ commutes with the operation P that implements $(x,k) \to (-x,-k)$. The simplicity of the eigenfunctions then implies that $P\varphi_j = \pm\varphi_j$. Hence the last term in the r.h.s. of (2.11), $\langle \varphi_j(\cdot;-k), \hat{V}_y(-k) \varphi_j(\cdot;-k) \rangle$ becomes

$$\begin{aligned} \int_{\mathbb{R}} \varphi_j(x;-k)^2 (-k - Bx) dx &= \int_{\mathbb{R}} \varphi_j(-x;-k)^2 (-k + Bx) dx \\ &= -\int_{\mathbb{R}} \varphi_j(x;k)^2 (k - Bx) dx \\ &= -\langle \varphi_j(\cdot;k), \hat{V}_y(k) \varphi_j(\cdot;k) \rangle, \end{aligned}$$

and the result follows from this, (2.11) and the Feynman-Hellmann formula,

$$\omega_j'(k) = 2 \langle \varphi_j(\cdot;k), (k - Bx) \varphi_j(\cdot;k) \rangle. \qquad (2.12)$$

■

One of the key points for the proof of Proposition 2.1 and for the estimation of the edge current given below, is the following Lemma. Its proof

relies on the fact (proved in Lemma 5.3 in Appendix 1) the dispersion curves $\omega_j(k)$, $j \in \mathbb{N}$, are separated, in the sense that

$$\inf_{k\in\mathbb{R}} |\omega_l(k) - \omega_j(k)| > 0,\ l \neq j,\ \text{for } 0 < C < \infty,$$

the same estimate being true if $C = +\infty$ by taking the infimum on any bounded set instead of $\mathbb{R}$.

Lemma 2.1 *Let the confining potential $V_0(x)$ be as in Proposition 2.1. Let*

$$\Delta_n \equiv [(2n+a)B, (2n+c)B],\ \textit{for } 1 < a < c < 3. \tag{2.13}$$

Then, for any $j, l = 0, 1, \ldots, n$, we have

$$\omega_j^{-1}(\Delta_n) \cap \omega_l^{-1}(\Delta_n) = \emptyset, \quad j \neq l, \tag{2.14}$$

provided $c - a$ is sufficiently small.

Proof.
Let us first consider the case $0 < C < \infty$. In light of Lemma 5.3, we know that

$$d_n \equiv \min_{0\le j\le n-1} \inf_{k\in\mathbb{R}} (\omega_{j+1}(k) - \omega_j(k)) > 0.$$

But any $k \in \omega_j^{-1}(\Delta_n) \cap \omega_l^{-1}(\Delta_n)$ satisfying $0 \le \omega_{j+1}(k) - \omega_j(k) < (c-a)B$, we see this inequality leads to a contradiction if $c - a < d_n/B$. As a consequence we have $\omega_j^{-1}(\Delta_n) \cap \omega_l^{-1}(\Delta_n) = \emptyset$ for any $j \neq l$ provided $|\Delta_n|/B < d_n$.

In the case where $C = +\infty$, we deduce from the evenness and the asymptotic behavior of ω_1 (see Lemmas 5.1 and 5.2 in Appendix 1) there is a real number $k_0 > 0$ such that for all $|k| > k_0$ we have

$$\omega_1(k) > (2n+c)B.$$

The result follows from this by arguing as before with

$$d_n \equiv \min_{0\le j\le n-1} \inf_{|k|\le k_0} (\omega_{j+1}(k) - \omega_j(k)) > 0.$$

■

2.2 Edge Currents for a Parabolic Confining Potential

As a warm up, we address now the model studied by Exner, Joye and Kovarik in [2], where the confining potential is given by (1.4). For this model, the electron is confined to a parabolic channel of infinite extent in the y-direction. For any $E > 0$, the plane $\mathbb{R}^2$ is divided into a classically allowed region given by $|x| < \sqrt{E/g}$, and the complementary classically forbidden region.

Let us define a modified field strength by $B_g \equiv \sqrt{B^2 + g^2}$. The reduced, unperturbed Hamiltonian for the parabolic channel problem is given by

$$\begin{aligned} h_0(k) &= p_x^2 + (k - Bx)^2 + g^2 x^2 \\ &= p_x^2 + \left(B_g x - \frac{B}{B_g} k \right)^2 + \left(\frac{g}{B_g} \right)^2 k^2. \end{aligned} \tag{2.15}$$

Since this is simply a shifted harmonic oscillator Hamiltonian, it is completely solvable. The dispersion curves have the following explicit expression

$$\omega_j(k) = (2j+1) B_g + \left(\frac{g}{B_g} \right)^2 k^2, \tag{2.16}$$

and the associated normalized eigenfunctions are given by

$$\varphi_j(x;k) = \frac{1}{\sqrt{2^j j!}} \left(\frac{B}{\pi} \right)^{1/4} e^{-B_g/2(x-(B/B_g^2)k)^2} H_j(\sqrt{B_g}(x - (B/B_g^2)k)), \tag{2.17}$$

where H_j is the j^{th} Hermite polynomial. The dispersion curves $\omega_j(k)$ being parabolas with equation (2.16), the set $\omega_j^{-1}(\Delta_n)$ for the interval

$$\Delta_n \equiv [(2n+a)B_g, (2n+c)B_g], \ 1 < a < c < 3, \tag{2.18}$$

is explicitly known:

$$\omega_j^{-1}(\Delta_n)_- = [-k_j^{(n)}(c), -k_j^{(n)}(a)], \tag{2.19}$$

with

$$k_j^{(n)}(x) \equiv \frac{B_g^{3/2}}{g} \sqrt{2(n-j) + x - 1}, \ x = a, c. \tag{2.20}$$

Henceforth,

$$-\omega_j'(k) = -2 \left(\frac{g}{B_g} \right)^2 k \geq 2 \left(\frac{g}{B_g} \right)^2 k_j^{(n)}(a),$$

for each $k \in \omega_j^{-1}(\Delta_n)_-$, which leads to

$$-\omega_j'(k) \geq 2\left(\frac{g}{\sqrt{B_g}}\right)\sqrt{2(n-j)+a-1},\ k \in \omega_j^{-1}(\Delta_n)_-, \tag{2.21}$$

according to (2.20).

Let us consider a state of finite energy ψ, with $\psi \in E_0(\Delta_n)L^2(\mathbb{R}^2)$, whose Fourier coefficients $\beta_j(k)$, $j = 0, 1, \ldots, n$, are defined as in (2.5). We assume in addition there is a constant $\gamma > 0$ such that the $\beta_j(k)$ satisfy the following condition:

$$|\beta_j(k)|^2 \geq (1+\gamma^2)|\beta_j(-k)|^2,\ k \in \omega_j^{-1}(\Delta_n)_-, j = 0, 1, \cdots, n. \tag{2.22}$$

Thus $|\beta_j(k)|^2 - |\beta_j(-k)|^2 \geq \gamma^2/(1+\gamma^2)|\beta_j(k)|^2$ for all $j = 0, 1, \ldots, n$ and $k \in \omega_j^{-1}(\Delta_n)_-$, so we get

$$\sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)_-} |\beta_j(k)|^2 dk \geq \frac{1+\gamma^2}{2+\gamma^2}\|\psi\|^2, \tag{2.23}$$

from the normalization condition (2.6). It follows readily from this and from the expression (2.8) of the total current carried by the state ψ,

$$\langle \psi, V_y \psi \rangle = \frac{1}{2}\sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)_-} (|\beta_j(k)|^2 - |\beta_j(-k)|^2)\omega_j'(k) dk,$$

together with the estimate (2.21), that

$$-\langle \psi, V_y \psi \rangle \geq \frac{\gamma^2}{2+\gamma^2}\sqrt{(a-1)}\frac{g}{B_g^{1/2}}. \tag{2.24}$$

Notice that the lower bound to the current in (2.24) is actually of size $B^{1/2}$ since g has the same dimension as B.

2.3 Estimation of the Edge Current for a Strip

We turn now to the estimation of the left-edge current for a strip of width $L > 0$. Namely, we assume the confining potential V_0 is an even function satisfying (2.7) and such that

$$V_0(x)\chi_{\{|x|<L/2\}}(x) = 0. \tag{2.25}$$

We want to estimate the total current along both edges, carried by appropriately chosen states ψ. That is, we want to obtain a lower bound on the matrix element of the localized velocity operator (2.8), carried by a state $\psi \in E_0(\Delta_n)L^2(\mathbb{R}^2)$ associated to the energy interval $\Delta_n \subset (E_n(B), E_{n+1}(B))$. Much of the technical work in this paper is devoted to bounding $(-\omega_j'(k))$ from below, uniformly for k in $\omega_j^{-1}(\Delta_n)_-$.

Lemma 2.2 *Let Δ_n be as in Lemma 2.1 and β_j, $j = 0, 1, \cdots, n$, be defined by (2.5). Then, there is a constant $C_n > 0$ independent of B such that*

$$-\omega_j'(k) \geq C_n(a-1)^2(3-c)^2 B^{1/2}, \; k \in \omega_j^{-1}(\Delta_n)_-, \tag{2.26}$$

provided B is large enough and V_0 satisfies one the two following conditions:

- *V_0 is the sharp confining potential defined by (1.2) and $\mathcal{V}_0 \geq 2(2n+c)B$,*
- *V_0 is a power function as in (1.3) and $\mathcal{V}_0 \geq (2n+c)B^{(p+2)/2}$.*

Moreover C_n does not depend on $\mathcal{V}_0$ in the case of the sharp confining potential, while $C_n = \tilde{C}_n/v$ where $v = (2n+c)B^{-(p+2)/2}\mathcal{V}_0 \geq 1$ and $\tilde{C}_n$ is independent of $\mathcal{V}_0$ for the power function confining potential.

Proof.
Inserting the commutator formula

$$\hat{V}_y(k) = (k - Bx) \equiv \frac{-i}{2B}[p_x, h_0(k)] + \frac{1}{2B}V_0' \tag{2.27}$$

for $\hat{V}_y(k)$ into (2.12), we obtain two terms. Due to the Virial Theorem, the term involving the commutator $[p_x, h_0(k)]$ vanishes as in the one-edge case, giving:

$$\omega_j'(k) = \frac{1}{2B}\langle \varphi_j(\cdot;k), V_0'\varphi_j(\cdot;k)\rangle. \tag{2.28}$$

The end of the proof also consists in bounding the remaining term $\langle \varphi_j(\cdot;k), V_0'\varphi_j(\cdot;k)\rangle$ from above by a (negative) constant times $B^{3/2}$. This technical computation is postponed to section 2.5 for the sharp confining potential and to section 2.6 for the power function confining potential. In both cases the technique used is based on Lemmas 2.3 and 2.4 given in section 2.4 below. ■

In light of (2.8) and Lemma 2.2, let us see now the current carried by a state ψ, whose coefficients $\beta_j(k)$, $j = 0, 1, \cdots, n$, are mostly supported on the set of negative wave numbers k, is of size $B^{1/2}$.

Theorem 2.1 *Let Δ_n and V_0 be as in Lemma 2.2, and ψ satisfy the condition (2.22): There is $\gamma > 0$ such that*

$$|\beta_j(k)|^2 \geq (1+\gamma^2)|\beta_j(-k)|^2, \;\; k \in \omega_j^{-1}(\Delta_n)_-, j = 0, 1, \cdots, n.$$

Then for sufficiently large B, we have

$$-\langle \psi, V_y \psi \rangle \geq \frac{\gamma^2}{2+\gamma^2} C_n (a-1)^2 (3-c)^2 B^{1/2} \|\psi\|^2, \tag{2.29}$$

where C_n is the constant defined in Lemma 2.2.

Proof.
By recalling the estimate (2.23),

$$\sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)_-} |\beta_j(k)|^2 dk \geq \frac{1+\gamma^2}{2+\gamma^2} \|\psi\|^2,$$

which derives from (2.22) together with the normalization condition (2.6), the result immediately follows from (2.8) and (2.26). ■

2.4 Bounding the Right Current Term

As in section 2.3 we assume the confining potential V_0 is an even function satisfying (2.7) and (2.25). This is the case for the step function confining potential (1.2) and the power function confining potential (1.3) we will consider below.

For any $j = 0, 1, \ldots, n$ it is clear from the definition of $\omega_j^{-1}(\Delta_n)_-$ that $\sup \omega_j^{-1}(\Delta_n)_- \leq 0$. Actually, we establish in Lemma 2.3 this supremum is bounded by a number arbitrarily close to $(-BL)/2$, provided the magnetic strength B is taken sufficiently large. Consequently, the region $x \geq 0$ is in the classically forbidden zone for energies $\omega_j(k)$, $k \in \omega_j^{-1}(\Delta_n)_-$, at least in the intense magnetic field regime. This is because the parabolic part of the effective potential

$$W_j(x; k) \equiv (k - Bx)^2 + V_0(x) - \omega_j(k), \tag{2.30}$$

is centered at the coordinate k/B.

Henceforth the eigenfunctions $\varphi_j(.;k)$ of $h_0(k)$ are exponentially decaying in the region $x \geq 0$ for all $k \in \omega_j^{-1}(\Delta_n)_-$. This is not true in the region $x \leq 0$. Due to the evenness of V_0, $\int_{\mathbb{R}_+} V_0'(x)\varphi_j(x;k)^2 dx$ is also expected to be small relative to $\int_{\mathbb{R}_-} V_0'(x)\varphi_j(x;k)^2 dx$, so

$$\omega_j'(k) \approx \frac{1}{2B} \int_{\mathbb{R}_-} V_0'(x)\varphi_j(x;k)^2 dx,$$

according (2.28). This remark is made precise below. Namely we state in Lemma 2.4 that the remaining term $\int_{\mathbb{R}_-} V_0'(x)\varphi_j(x;k)^2 dx$ is bounded by a constant times B and we establish in sections 2.5-2.6 for the step function confining potential (1.2) and the power function confining potential (1.3), the main term $\int_{\mathbb{R}_-} V_0'(x)\varphi_j(x;k)^2 dx$ is of size $B^{3/2}$.

Wave Numbers Estimate

Lemma 2.3 *Let Δ_n be as in Lemma 2.1 and V_0 be an even function satisfying (2.7) and (2.25). Then, any given $\alpha > 2$, there is $B_\alpha \geq 1$ such that*

$$\omega_j^{-1}(\Delta_n)_- \subset (-\infty, -BL/\alpha),$$

for all $B \geq B_\alpha$ and $\mathcal{V}_0 > 0$.

Proof.
Let θ_ϵ be a real valued, even and twice continuously differentiable function in $\mathbb{R}$, such that

$$\theta_\epsilon(x) = \begin{cases} 1 & \text{if } -L/2 + \epsilon/2 \leq x \leq 0 \\ 0 & \text{if } x < -L/2 + \epsilon/4, \end{cases}$$

for some ϵ in $(0, L/2)$. The function $\theta_\epsilon(x)\psi_n(x;k)$ (where $\psi_n(x;k)$ still denotes the n^{th} normalized eigenfunction of $h_L(k) = p_x^2 + (k - Bx)^2)$ obviously belongs to the domain of $h_0(k)$. Moreover, the supports of V_0 and θ_ϵ being disjoint, the following identity holds true:

$$\begin{aligned} (h_0(k) - (2n+1)B)\theta_\epsilon(x)\psi_n(x;k) &= [h_0(k), \theta_\epsilon]\psi_n(x;k) \\ &= -(\theta_\epsilon'' + 2i\theta_\epsilon' p_x)\psi_n(x;k). \end{aligned}$$

This immediately entails:

$$\|(h_0(k) - (2n+1)B)\theta_\epsilon\psi_n(.;k)\| \leq \|\theta_\epsilon''\psi_n(.;k)\| + 2\|\theta_\epsilon' p_x\psi_n(x;k)\|. \quad (2.31)$$

Let us suppose now that $k/B \in [-L/2+\epsilon, 0]$. Then, using the explicit expression (2.46) of $\psi_n(.;k)$ and bearing in mind the vanishing of θ'_ϵ outside $[-L/2+\epsilon/4, -L/2+\epsilon/2] \cup [L/2-\epsilon/2, L/2-\epsilon/4]$, it is possible to find two constants α_n and β_n independent of B, $\mathcal{V}_0$ and ϵ, such that:

$$\begin{cases} \|\theta''_\epsilon \psi_n(.;k)\| \leq \alpha_n \epsilon^{-3/2} B^{1/4} \mathrm{e}^{-B/8\epsilon^2} \\ \|\theta'_\epsilon p_x \psi_n(x;k)\| \leq \beta_n \epsilon^{-1/2} B^{1/4} (B^{1/2}+\epsilon B) \mathrm{e}^{-B/8\epsilon^2}. \end{cases}$$

This, combined with (2.31), involves

$$\|(h_0(k)-(2n+1)B)\theta_\epsilon \psi_n(.;k)\| \leq \gamma_n B^{1/4} \epsilon^{-3/2} (1+\epsilon B^{1/2}+\epsilon^2 B) \mathrm{e}^{-B/8\epsilon^2}, \quad (2.32)$$

where we have set $\gamma_n = \alpha_n + 2\beta_n$.
Next, by performing the change of variable $y = B^{1/2}(x-k/B)$ in the integral $\int_{\mathbb{R}} \theta_\epsilon^2 \psi_n(x;k)^2 dx$ we get

$$\begin{aligned} 2^n n! \sqrt{\pi} \|\theta_\epsilon \psi_n(.;k)\|^2 &\geq \int_{B^{1/2}(-L/2+\epsilon/2-k/B)}^{B^{1/2}(L/2-\epsilon/2-k/B)} H_n(y)^2 \mathrm{e}^{-y^2} dy \\ &\geq \int_0^{L/4} H_n(y)^2 \mathrm{e}^{-y^2} dy > 0, \end{aligned}$$

for all $B \geq 1$. In light of (2.32), we see there is also a constant C_n independent of B, $\mathcal{V}_0$ and ϵ such that

$$\operatorname{dist}\,(\sigma(h_0(k)), (2n+1)B) \leq C_n B^{1/4} \epsilon^{-3/2} (1+\epsilon B^{1/2}+\epsilon^2 B) \mathrm{e}^{-B/8\epsilon^2},$$

provided B is sufficiently large. This, combined with the simplicity of the $\omega_m(k)$, entails

$$\omega_n(k) \leq (2n+1)B + C_n B^{1/4} \epsilon^{-3/2} (1+\epsilon B^{1/2}+\epsilon^2 B) \mathrm{e}^{-B/8\epsilon^2},$$

proving that $\omega_n(k)$ can be made smaller than $(2n+a)B$ by taking B sufficiently large. Hence we have shown that

$$\omega_j(k) \notin \Delta_n, \ j = 0, 1, \ldots, n, \ k/B \in [-L/2+\epsilon, 0], \quad (2.33)$$

and the result follows from (2.33) for all $k \in [-BL/\alpha, 0]$ by taking $\epsilon = (\alpha-2)/(2\alpha)L$. ■

Trace Function Estimate in the Classically Forbidden Zone

The main consequence of the preceding Lemma is the positivity of the effective potential $W_j(x;k)$ defined by (2.30) for $k \in \omega_j^{-1}(\Delta_n)_-$ in the region $x \geq 0$. Indeed, we know from Lemma 2.3 we can make B large enough so $\omega_j^{-1}(\Delta_n)_- \subset (-\infty, -BL/3)$, and consequently $W_j(x;k) \geq B^2L^2/36 - (2n+c)B$ for all $x \geq -L/6$ and $k \in \omega_j^{-1}(\Delta_n)_-$. Whence there is necessarily $B_0 > 0$ such that:

$$W_j(x;k) \geq \left(\frac{BL}{8}\right)^2 > 0,\ x \geq -L/6,\ k \in \omega_j^{-1}(\Delta_n)_-,\ B \geq B_0. \tag{2.34}$$

The eigenfunction $\varphi_j(.;k)$ being an $H^1(\mathbb{R})$-solution to the differential equation $\varphi''(x) = W_j(x;k)\varphi(x)$, is also exponentially decaying in the region $x \geq -L/6$. Namely, we have

$$0 \leq \varphi_j(t;k) \leq \varphi_j(s;k)\mathrm{e}^{-\int_s^t \sqrt{W_j(x;k)}dx},\ -L/6 \leq s \leq t, \tag{2.35}$$

from Proposition 8.2 in [8]. This estimate is the main tool to bound $\int_{\mathbb{R}_+} V_0'(x)\varphi_j(x;k)^2dx$ as in Lemma 2.4. The proof consists in relating this integral to $\int_{\mathbb{R}_+}(Bx - k)\varphi_j(x;k)^2dx$ through the generalized expression (2.40) of the Feynman-Hellmann relation. Concerning, $\int_{\mathbb{R}_+}(Bx-k)\varphi_j(x;k)^2dx$, upon choosing B is sufficiently large, we actually have:

$$\int_0^{+\infty}(Bx-k)\varphi_j(x;k)^2dx \leq \frac{BL}{2}\mathrm{e}^{-BL^2/24},\ k \in \omega_j^{-1}(\Delta_n)_-. \tag{2.36}$$

The proof of (2.36) is based on the estimate (2.35) and consists in 2 steps.
First Step. In light of Lemma 2.3, we choose B large enough so

$$0 \leq Bt - k \leq 2W_j^{1/2}(t;k),\ t \geq 0,\ k \in \omega_j^{-1}(\Delta_n)_-,$$

then we combine this estimate with (2.35) written with $s = 0$ and integrate the obtained inequality over $(0, +\infty)$, getting:

$$\int_0^{+\infty}(Bt-k)\varphi_j(t;k)^2dt \leq \varphi_j(0;k)^2. \tag{2.37}$$

Second Step. We insert (2.34) in (2.35) written with $t = 0$, square,

$$\varphi_j(0;k)^2\mathrm{e}^{-(BL/4)s} \leq \varphi_j(s;k)^2,\ -L/6 \leq s \leq 0,$$

then we integrate the obtained inequality with respect to s over the interval $(-L/6, 0)$. Thus, using the normalization condition $\|\varphi_j(.;k)\| = 1$, we obtain:

$$\varphi_j(0;k)^2 \leq \frac{BL}{2}\mathrm{e}^{-BL^2/24},\ k \in \omega_j^{-1}(\Delta_n)_-. \tag{2.38}$$

Finally (2.36) follows from (2.37) and (2.38).

Armed with (2.36) we turn now to establishing the main result of this section.

Bounding the Right Current Term

Lemma 2.4 *Let Δ_n and V_0 be as in Lemma 2.3. Then, there is $B_1 > 0$ and a constant $\gamma(n,j) > 0$ independent of B and $\mathcal{V}_0$, such that:*

$$\int_{\mathbb{R}_+} V_0'(x)\varphi_j(x;k)^2 dx \leq \gamma(n,j)B,\ k \in \omega_j^{-1}(\Delta_n)_-,\ B \geq B_1. \tag{2.39}$$

Proof.
Let $\rho \in C^3(\mathbb{R})$ be a bounded real-valued function and A denote the selfadjoint operator $\rho(x)p_x + p_x\rho(x)$ in $L^2(\mathbb{R})$, with domain $H^1(\mathbb{R})$. Any function φ in the domain of $h_0(k)$ belonging to $H^1(\mathbb{R})$, $\langle [A, h_0(k)]\varphi, \varphi\rangle_{L^2(\mathbb{R})}$ can be defined as $\langle h_0(k)\varphi, A\varphi\rangle_{L^2(\mathbb{R})} - \langle A\varphi, h_0(k)\varphi\rangle_{L^2(\mathbb{R})}$, and standard computations provide:

$$\begin{aligned}\langle -i[A, h_0(k)]\varphi, \varphi\rangle_{L^2(\mathbb{R})} &= +4\langle \rho'\varphi', \varphi'\rangle_{L^2(\mathbb{R})} - 4B\langle \rho(Bx-k)\varphi, \varphi\rangle_{L^2(\mathbb{R})} \\ &\quad -2\langle \rho V_0'\varphi, \varphi\rangle_{L^2(\mathbb{R})} - \langle \rho'''\varphi, \varphi\rangle_{L^2(\mathbb{R})}. \end{aligned} \tag{2.40}$$

Here $\langle \rho V_0'\varphi, \varphi\rangle_{L^2(\mathbb{R})}$ means $\mathcal{V}_0(\rho(L/2)\varphi^2(L/2) - \rho(-L/2)\varphi^2(-L/2))$ in the case where V_0 is the sharp confining potential (1.2). When φ is an eigenfunction $\varphi_j(.;k)$ of $h_0(k)$, the scalar product $\langle -i[A, h_0(k)]\varphi, \varphi\rangle_{L^2(\mathbb{R})}$ vanishes according to the Virial Theorem. Henceforth by taking ρ such that $\rho(x) = 0$ if $x \leq 0$, and $\rho(x) = 1$ if $x \geq L/2$, we deduce from (2.40)

$$\begin{aligned} & 2\langle \rho V_0'\varphi_j(.;k), \varphi_j(.;k)\rangle_{L^2(\mathbb{R})} \\ \leq\ & \|\rho'''\|_\infty \int_0^{L/2} \varphi_j(x;k)^2 dx + 4B\|\rho\|_\infty \int_0^\infty (Bx-k)\varphi_j(x;k)^2 dx \\ & +4\|\rho'\|_\infty \int_0^{L/2} \varphi_j'(x;k)^2 dx. \end{aligned}$$

Hence the result follows from this, (2.36), together with the basic inequality $\int_{\mathbb{R}} \varphi_j'(x;k)^2 dx \leq (2n+c)B$. ■

Notice that (2.39) actually reduces to

(i)	$\mathcal{V}_0 \varphi_j(L/2;k)^2 \leq \gamma(n,j)B$	if V_0 is given by (1.2)
(ii)	$\mathcal{V}_0 \int_{(L/2,+\infty)} (x-L/2)^{p-1} \varphi_j(x;k)^2 dx \leq \gamma(n,j)B$	if V_0 is given by (1.3).

These two estimates are useful for the two following sections.

2.5 The Sharp Confining Potential

The sharp confining potential V_0, defined by (1.2) confines particles with energy less than $\mathcal{V}_0$ to the strip $-L/2 \leq x \leq L/2$. For this model, we have

$$V_0'(x) = \mathcal{V}_0(\delta(x+L/2) - \delta(x-L/2))$$

in the distributional sense, so the derivative of the j^{th} dispersion curve can be expressed as

$$\omega_j'(k) = -\frac{\mathcal{V}_0}{2B} \left(\varphi_j(-L/2;k)^2 - \varphi_j(L/2;k)^2 \right), \tag{2.41}$$

according to (2.28).

The case of the left part of the current is treated by Lemma 2.4: Upon taking B sufficiently large we have,

$$\mathcal{V}_0 \varphi_j(L/2;k)^2 \leq \gamma(n,j)B, \ k \in \omega_j^{-1}(\Delta_n)_-, \tag{2.42}$$

the constant $\gamma(n,j) > 0$ being independent of B and $\mathcal{V}_0$.

We turn now to computing a lower bound on the trace term $\mathcal{V}_0 \varphi_j(-L/2;k)^2$. This will require several steps.

Step 1 : Harmonic Oscillator Eigenfunction Comparison Revisited The proof of Lemma 2.2 in [8] (based on the properties of the eigenfunctions $\psi_m(.;k)$ of the harmonic oscillator $h_L(k) = p_x^2 + (Bx-k)^2$) applying without change to the case of the strip geometry examined here, the following estimate,

$$|\langle \varphi_j(.;k), V_0 P_n \varphi_j(.;k) \rangle| \geq \frac{1}{2(n+1)B} (\omega_j(k) - E_n(B))(E_{n+1}(B) - \omega_j(k)), \tag{2.43}$$

holds for all $k \in \omega_j^{-1}(\Delta_n)_-$. We recall that P_n denotes the projection on the eigenspace spanned by the first n eigenfunctions $\psi_m(.;k)$ of $h_L(k)$,

$$P_n\varphi_j(x;k) \equiv \sum_{m=0}^{n} \alpha_m^{(j)}(k)\psi_m(x;k), \tag{2.44}$$

with

$$\alpha_m^{(j)}(k) \equiv \langle \varphi_j(.;k), \psi_m(.;k)\rangle, \tag{2.45}$$

and that the explicit expression of $\psi_m(x;k)$ is

$$\psi_m(x;k) = \frac{1}{\sqrt{2^m m!}}\left(\frac{B}{\pi}\right)^{1/4} H_m(\sqrt{B}(x-k/B))\mathrm{e}^{-B/2(x-k/B)^2}, \tag{2.46}$$

where H_m denotes the m^{th} Hermite polynomial function as in [8].

The strategy consists in computing an upper bound on $|\langle \varphi_j(.;k), V_0 P_n \varphi_j(.;k)\rangle|$, involving the trace $\mathcal{V}_0^2\varphi_j(-L/2;k)^2$. To do that, we first calculate the scalar product $\langle \varphi_j(.;k), V_0 P_n \varphi_j(.;k)\rangle$ by expanding $P_n\varphi_j(.;k)$ as in (2.44):

$$|\langle \varphi_j(.;k), V_0 P_n \varphi_j(.;k)\rangle| \leq \mathcal{V}_0 \sum_{m=0}^{n} |\alpha_m^{(j)}(k)| \int_{|x|\geq L/2} |\varphi_j(x;k)||\psi_m(x;k)|dx. \tag{2.47}$$

For this model, the set $|x| > L/2$ is the classically forbidden region for electrons with energy less than $\mathcal{V}_0$, so

$$0 \leq \varphi_j(x;k) \leq \varphi_j(\pm L/2;k)\mathrm{e}^{\mp\sqrt{\mathcal{V}_0-\omega_j(k)}(x\mp L/2)},\ \pm x \geq L/2,$$

from Proposition 8.3 in [8]. Henceforth by substituting the corresponding exponentially decreasing term for $\varphi_j(.;k)$ in (2.47), we obtain

$$\begin{aligned} &|\langle \varphi_j(.;k), V_0 P_n \varphi_j(.;k)\rangle| \\ \leq\ & \mathcal{V}_0 \sum_{m=0}^{n} |\alpha_m^{(j)}(k)| \Big((I_{m,-}^{(j)})\varphi_j(-L/2;k) + (I_{m,+}^{(j)})\varphi_j(L/2;k)\Big), \end{aligned} \tag{2.48}$$

where we have set

$$I_{m,\pm}^{(j)} \equiv \int_{\pm x\geq L/2} |\psi_m(x;k)|\mathrm{e}^{\mp\sqrt{\mathcal{V}_0-\omega_j(k)}(x\mp L/2)}dx. \tag{2.49}$$

Step 2 : Trace Function Estimate
In view of bounding the integrals $I_{m,\pm}^{(j)}$ we first define the constant

$$\mathcal{H}_m \equiv \sup_{u\in\mathbb{R}} H_m(u)\mathrm{e}^{-u^2/2}. \tag{2.50}$$

Then we substitute the following estimate

$$|\psi_m(x;k)| \leq \left(\frac{B}{\pi}\right)^{1/4} \frac{\mathcal{H}_m}{\sqrt{2^m m!}}, \tag{2.51}$$

which obviously follows from (2.46) and (2.50), for $|\psi_m(x;k)|$ in (2.49), and get:

$$I_{m,\pm}^{(j)} \leq \left(\frac{B}{\pi}\right)^{1/4} \frac{\mathcal{H}_m}{\sqrt{2^m m!}} \frac{1}{\sqrt{\mathcal{V}_0 - \omega_j(k)}}. \tag{2.52}$$

Now combining (2.48) with (2.52), we obtain

$$\begin{aligned} & |\langle \varphi_j(.,k), V_0 P_n \varphi_j(.,k)\rangle_{L^2(\mathbb{R}^2)}| \\ \leq \; & \frac{\mathcal{V}_0}{\sqrt{\mathcal{V}_0 - \omega_j(k)}} \left(\frac{B}{\pi}\right)^{1/4} \left(\sum_{m=0}^{n} \frac{\mathcal{H}_m}{\sqrt{2^m m!}} |\alpha_m^{(j)}(k)|\right) (\varphi_j(-L/2;k) + \varphi_j(L/2;k)). \end{aligned} \tag{2.53}$$

Let us define the constant $\mathcal{H}^{(n)}$ by

$$\mathcal{H}^{(n)} \equiv \left(\sum_{m\leq n} \frac{\mathcal{H}_m^2}{2^m m!}\right)^{1/2}. \tag{2.54}$$

Then we apply the Cauchy-Schwarz inequality to the sum in (2.53), and use the normalization condition

$$\sum_{m=0}^{n} |\alpha_m^{(j)}(k)|^2 = \|P_n \varphi_j(\cdot;k)\|^2 \leq 1,$$

so we end up getting:

$$\begin{aligned} & |\langle \varphi_j(.,k), V_0 P_n \varphi_j(.,k)\rangle_{L^2(\mathbb{R}^2)}| \\ \leq \; & \frac{\mathcal{V}_0}{\sqrt{\mathcal{V}_0 - \omega_j(k)}} \left(\frac{B}{\pi}\right)^{1/4} \mathcal{H}^{(n)} (\varphi_j(-L/2;k) + \varphi_j(L/2;k)). \end{aligned} \tag{2.55}$$

Thus (2.55) combined with (2.42) and (2.43) provides

$$\begin{aligned}
& \mathcal{V}_0^{1/2}\varphi_j(-L/2;k) \\
\geq \quad & \frac{\pi^{1/4}}{2(n+1)\mathcal{H}^{(n)}}\left(1-\frac{\omega_j(k)}{\mathcal{V}_0}\right)^{1/2}(a-1)(3-c)B^{3/4}-\sqrt{\alpha(n,j)}B^{1/4},
\end{aligned}$$

so there is a constant $D(n,j)$ independent of B and $\mathcal{V}_0$ such that

$$\mathcal{V}_0^{1/2}\varphi_j(-L/2;k)\geq D(n,j)(a-1)(3-c)B^{3/4}, \tag{2.56}$$

for all $j=0,1,\cdots,n$ and $k\in\omega_j^{-1}(\Delta_n)_-$. This estimate holds provided B is sufficiently large, $\mathcal{V}_0\geq 2(2n+c)B$ and $|\Delta_n|$ is small enough.

Now, the bound (2.26) on the derivative $\omega_j'(k)$ follows from (2.41), (2.42) and (2.56).

2.6 The Power Function Confining Potential

The second case we consider is the one for which the confining potential is a power function of x alone and is given by (1.3). Due to (2.28), the derivative of the j^{th} dispersion curve, $j=0,1,\ldots,n$, in the particular case of the power function confining potential (1.3) has the following expression

$$\omega_j'(k)=-p\frac{\mathcal{V}_0}{2B}\left(I_p^{(j)}(k)-I_p^{(j)}(-k)\right), \tag{2.57}$$

where

$$I_p^{(j)}(k)=\int_{-\infty}^{-L/2}(-x-L/2)^{p-1}\varphi_j(x;k)^2dx. \tag{2.58}$$

Here we used the symmetry property $\varphi_j(-x;k)^2=\varphi_j(x;-k)^2$ of the eigenfunctions established in Lemma 5.1.

The second term in (2.57) is treated by Lemma 2.4: There is a constant $\gamma(n,j)>0$ independent of B and $\mathcal{V}_0$ such that we have

$$0\leq\mathcal{V}_0 I_p^{(j)}(-k)\leq\gamma(n,j)B,\ \ k\in\omega_j^{-1}(\Delta_n)_-. \tag{2.59}$$

Actually (2.59) holds true provided B is taken sufficiently large.

We turn now to estimating from below the integral $I_p^{(j)}(k)$ defined in (2.58). We follow the calculation of section 2.5.

Step 1 : Harmonic Oscillator Eigenfunction Comparison
As in section 2.5 the starting point of the method is the estimate (2.43). Namely, for any $k \in \omega_j^{-1}(\Delta_n)$, we have

$$|\langle \varphi_j(\cdot;k), V_0 P_n \varphi_j(\cdot;k)\rangle| \geq \frac{1}{2B(n+1)}(\omega_j(k) - E_n(B))(E_{n+1}(B) - \omega_j(k)),$$

where P_n still denotes the projection on the eigenspace spanned by the first n eigenfunctions $\psi_m(.;k)$ of the harmonic oscillator Hamiltonian $h_L(k) = p_x^2 + (Bx-k)^2$. The strategy consists in computing an upper bound on $|\langle \varphi_j(\cdot;k), V_0 P_n \varphi_j(\cdot;k)\rangle|$, involving $\mathcal{V}_0 I_p^{(j)}(k)$. To do that, we expand $P_n\varphi_j(.;k)$ as in (2.44) in $\langle \varphi_j(\cdot;k), V_0 P_n \varphi_j(\cdot;k)\rangle$, getting (2.47). Then we substitute (1.3) for V_0 in (2.47) and obtain

$$|\langle \varphi_j(.;k), V_0 P_n \varphi_j(.;k)\rangle| \leq \mathcal{V}_0 \sum_{m=0}^{n} |\alpha_m^{(j)}(k)| \left(I_{p,m,-}^{(j)}(k) + I_{p,m,+}^{(j)}(k) \right), \quad (2.60)$$

where

$$I_{p,m,\pm}^{(j)}(k) \equiv \int_{\pm x \geq L/2} (\pm x - L/2)^p \, |\varphi_j(x;k)||\psi_m(x;k)| dx. \quad (2.61)$$

We are also left with the task of computing an upper bound for $I_{p,m,\pm}^{(j)}(k)$.

Step 2 : Integral Estimates.
1. Let k be in $\omega_j^{-1}(\Delta_n)_-$. By applying Lemma 2.3 once more we can choose the magnetic strength B sufficiently large so the quadratic potential

$$Q_m(x;k) \equiv (Bx-k)^2 - (2m+1)B \quad (2.62)$$

is positive in the region $x \geq 0$. Consequently the eigenfunction $\psi_m(.;k)$ of $h_L(k)$ decays exponentially in the region $x \geq 0$ since this is an $H^1(\mathbb{R})$-solution to the differential equation $\psi''(x) = Q_m(x;k)\psi(x)$. It follows from this (see Lemma 6.1 in Appendix 2) that:

$$0 \leq \mathcal{V}_0 I_{p,m,+}^{(j)}(k) \leq \frac{2}{L}\sqrt{(2n+c)B}, \; m = 0, 1, \ldots, n.$$

Next combining this estimate with the normalization condition $\sum_{m=0}^{n} |\alpha_m^{(j)}(k)|^2 \leq 1$, leads to

$$0 \leq \mathcal{V}_0 \sum_{m=0}^{n} |\alpha_m^{(j)}(k)| I_{p,m,+}^{(j)} \leq \frac{2}{L}\sqrt{(2n+c)nB}. \quad (2.63)$$

2. We turn now to computing an upper bound involving $I_p^{(j)}(k)$ on the integral

$$I_{p,m,-}^{(j)}(k) = \int_{-\infty}^{-L/2} (-x - L/2)^p \varphi_j(x;k)\psi_m(x;k)dx.$$

This can be made by applying the Cauchy-Schwartz inequality

$$I_{p,m,-}^{(j)}(k) \leq \left(\int_{-\infty}^{-L/2} (-x - L/2)^{p+1} \psi_m(x;k)^2 dx \right)^{1/2} I_p^{(j)}(k)^{1/2}, \qquad (2.64)$$

and bounding the prefactor $\int_{-\infty}^{-L/2} (-x - L/2)^{p+1} \psi_m(x;k)^2 dx$ as in Lemma 6.3 in Appendix 2. Namely, we assume that

$$\mathcal{V}_0 \geq (2n + c) B^{\frac{p+2}{2}}, \qquad (2.65)$$

so there is a constant $C_m^-(n,p)$ independent of B such that:

$$0 \leq \int_{-\infty}^{-L/2} (-x - L/2)^{p+1} \psi_m(x;k)^2 dx \leq C_m^-(n,p)^2 B^{-\frac{p+1}{2}}.$$

This, together with (2.64) involves

$$I_{p,m,-}^{(j)}(k) \leq C_m^-(n,p) B^{-\frac{p+1}{4}} I_p^{(j)}(k)^{1/2},$$

so we get

$$\sum |\alpha_m^{(j)}(k)| I_{p,m,-}^{(j)}(k) \leq C^-(n,p) B^{-\frac{p+1}{4}} I_p^{(j)}(k)^{1/2}, \qquad (2.66)$$

from the Cauchy-Schwartz inequality and the normalization condition $\sum_{m=0}^n |\alpha_m^{(j)}(k)|^2 \leq 1$, the constant $C^-(n,p)$ being defined as

$$C^-(n,p) \equiv \left(\sum_{m=0}^n (C_m^-(n,p))^2 \right)^{1/2}.$$

Step 3 : Estimate on the Main Term
By combining now the estimates (2.43), (2.60), (2.63) and (2.66), we end up getting:

$$\begin{aligned} & \mathcal{V}_0^{1/2} I_p^{(j)}(k)^{1/2} \\ \geq \quad & \frac{B^{\frac{p+5}{4}}}{C^-(n,p)\mathcal{V}_0^{1/2}} \left(\frac{(a-1)(3-c)}{2(n+1)} - \frac{2}{L}\sqrt{(2n+c)n} B^{-1/2} \right). \end{aligned}$$

This estimate remains valid as long as (2.65) holds true and B is sufficiently large. Whence there is a constant $C(n,p) > 0$ independent of B such that

$$\mathcal{V}_0 I_p^{(j)}(k) \geq \frac{C(n,p)}{v}(a-1)^2(3-c)^2 B^{\frac{3}{2}},\ k \in \omega_j^{-1}(\Delta_n),\ j = 0, 1, \cdots, n, \quad (2.67)$$

provided $\mathcal{V}_0$ has the following expression:

$$\mathcal{V}_0 = v(2n+c)B^{\frac{p+2}{2}}. \quad (2.68)$$

Here the coupling constant v is taken in $[1, +\infty)$ so the condition (2.65) is automatically satisfied with this choice of $\mathcal{V}_0$.

Now it is easy to check the bound (2.26) on the derivative $\omega_j'(k)$ follows from (2.57), (2.59) and (2.67).

2.7 Perturbation of Edge Currents

We now consider the perturbation of the edge currents by adding a bounded impurity Potential $V_1(x,y)$ to H_0. As in section 2.3 of [8] for unbounded geometries, we prove that the lower bound on the edge currents is stable with respect to these perturbations provided $\|V_1\|_\infty$ is not too large compared with B. We continue to use the same notation as in [8]. That is, $\Delta_n \subset \mathbb{R}$ denotes a closed, bounded interval with $\Delta_n \subset (E_n(B), E_{n+1}(B))$, for some $n \geq 0$. We can write the interval Δ_n as in (2.13):

$$\Delta_n = [(2n+a)B, (2n+c)B], \quad \text{for } 1 < a < c < 3. \quad (2.69)$$

We consider a larger interval $\tilde{\Delta}_n$ containing Δ_n, and with the same midpoint $E \equiv (2n + (a+c)/2)B$, and of the form

$$\tilde{\Delta}_n = [(2n+\tilde{a})B, (2n+\tilde{c})B], \quad \text{for } 1 < \tilde{a} < a < c < \tilde{c} < 3. \quad (2.70)$$

Theorem 2.2 *Let Δ_n and V_0 be as in Lemma 2.2. Let $V_1(x,y)$ be a bounded potential and let $E(\Delta_n)$ be the spectral projection for $H = H_0 + V_1$ and the interval Δ_n. Let $\psi \in L^2(\mathbb{R}^2)$ be a state satisfying $\psi = E(\Delta_n)\psi$. Let $\phi \equiv E_0(\tilde{\Delta}_n)\psi$ and $\xi \equiv E_0(\tilde{\Delta}_n^c)\psi$, so that $\psi = \phi + \xi$. Let ϕ have an expansion as in (2.4) with coefficients $\beta_j(k)$ satisfying the condition (2.22) of Theorem 2.1, that is:*

$$\exists \gamma > 0,\ |\beta_j(k)|^2 \geq (1+\gamma^2)|\beta_j(-k)|^2,\ k \in \omega_j^{-1}(\Delta_n)_-, j = 0, 1, \cdots, n.$$

Then, we have,

$$-\langle\psi, V_y\psi\rangle \geq B^{1/2}\left(\frac{\gamma^2}{2+\gamma^2}C_n(3-\tilde{c})^2(\tilde{a}-1)^2 - F_n(\|V_1\|/B)\right)\|\psi\|^2, \quad (2.71)$$

where $C_n > 0$ *is the constant defined in Lemma 2.2 and*

$$\begin{aligned} & F_n(\|V_1\|/B) \\ = & \left(\frac{2}{\tilde{c}-\tilde{a}}\right)^{1/2}\left(\frac{c-a}{2}+\frac{\|V_1\|}{B}\right)^{1/2} \times \left[2\left(2n+c+\frac{\|V_1\|}{B}\right)^{1/2}\right. \\ & \left. +\frac{\gamma^2}{2+\gamma^2}C_n(3-\tilde{c})^2(\tilde{a}-1)^2\left(\frac{2}{\tilde{c}-\tilde{a}}\right)^{3/2}\left(\frac{c-a}{2}+\frac{\|V_1\|}{B}\right)^{3/2}\right] \end{aligned} \quad (2.72)$$

If we suppose that $\|V_1\|_\infty < v_1 B$*, then for a fixed level* n*, if* $c-a$ *and* v_1 *are sufficiently small (depending on* $\tilde{a}$*,* $\tilde{c}$*, and* n*), there is a constant* $D_n > 0$ *so that for all* B*, we have*

$$-\langle\psi, V_y\psi\rangle \geq D_n B^{1/2}\|\psi\|. \quad (2.73)$$

Proof.
With reference to the definitions (2.69) and (2.70), we write the function ψ as

$$\psi = E_0(\tilde{\Delta}_n)\psi + E_0(\tilde{\Delta}_n^c)\psi \equiv \phi + \xi. \quad (2.74)$$

Next we use the selfadjointness of V_y in $L^2(\mathbb{R}^2)$, to write

$$\begin{aligned} \langle\psi, V_y\psi\rangle &= \langle\phi, V_y\phi\rangle \\ &+ \langle\psi, V_y\xi\rangle + \langle V_y\xi, \phi\rangle, \end{aligned} \quad (2.75)$$

whence, by using the Cauchy-Schwartz inequality, we obtain

$$-\langle\psi, V_y\psi\rangle \geq -\langle\phi, V_y\phi\rangle - 2\|V_y\xi\|_{L^2(\mathbb{R}^2)}\|\psi\|. \quad (2.76)$$

The result follows from Theorem 2.1 provided we have a good bound on $\|\xi\|$ and on $\|V_y\xi\|$. We recall from section 2.3 in [8] that

$$\|\xi\| \leq \left(\frac{2}{\tilde{c}-\tilde{a}}\right)\left(\frac{c-a}{2}+\frac{\|V_1\|}{B}\right)\|\psi\|, \quad (2.77)$$

and

$$\|V_y\xi\|^2 \leq \langle \xi, H_0 \xi\rangle \leq ((2n+c)B + \|V_1\|)\ \|\xi\|\ \|\psi\|. \tag{2.78}$$

The lower bound on the main term in (2.76) follows from the estimate (2.29):

$$\begin{aligned} -\langle \phi, V_y \phi\rangle &\geq \frac{\gamma^2}{2+\gamma^2} C_n (\tilde{a}-1)^2 (\tilde{c}-3)^2\ B^{1/2} \left(\sum_{j=0}^{n} \int_{\omega_j^{-1}(\tilde{\Delta}_n)} |\beta_j(k)|^2\ dk \right) \\ &\geq \frac{\gamma^2}{2+\gamma^2} C_n (\tilde{a}-1)^2 (\tilde{c}-3)^2\ B^{1/2} (\|\psi\|^2 - \|\xi\|^2), \end{aligned} \tag{2.79}$$

since

$$\sum_{j=0}^{n} \int_{\omega_j^{-1}(\tilde{\Delta}_n)} |\beta_j(k)|^2\ dk = \|\phi\|^2 = \|\psi\|^2 - \|\xi\|^2.$$

Combining this lower bound (2.79), with the estimate on $\|\xi\|$ in (2.77), and $\|V_y \xi\|$ in (2.78), we find (2.71) with the constant (2.72). This completes the proof. ■

3 Two-Edge Geometries: Spectral Properties and the Mourre Estimate

We now examine the spectral properties of the Hamiltonian $H = H_0 + V_1$, for suitable perturbations V_1, for two-edge geometries, paralleling the study in sections 4 and 5 of [8] for one-edge geometries. We use the commutator method of Mourre [1, 10]. For two-edge geometries, an analysis of the dispersion curves for H_0 showed that $\omega_j'(k)$ does not have fixed sign. Consequently, the local commutator used for the one-edge geometries in section 2.5, does not immediately apply. We first construct an appropriate conjugate operator S_α for H_0 with a general confining potential $V_0(x)$. By standard arguments [1], this proves the existence of absolutely continuous spectrum of H_0 at energies away from the Landau levels for sufficiently large B. Of course, the spectral properties of H_0 can be obtained directly from the direct integral decomposition (2.1) and an analysis of the spectrum of $h_0(k)$ defined by (2.2). This proves that the spectrum of H_0 is everywhere purely absolutely continuous. The advantage of the Mourre method, however, is that we can obtain the stability of the absolutely continuous spectrum between Landau

levels under two classes of perturbations V_1. We prove that the spectrum of H is purely absolutely continuous if 1) $V_1(x, y)$ is periodic with respect to y with sufficiently small period or 2) $V_1(x, y)$ has some decay in y-direction. These results are similar to those of Exner, Joye, and Kovarik [2]. We point out that for the more general class of perturbations V_1 treated in sections 4 and 5 of [8], such as random potentials, we do not know the spectral type of the operator H. However, we still know that there are states carrying nontrivial edge currents. As follows from the work of Ferrari and Macris [4, 5], the existence of edge currents is not tied to the spectral properties of H. Indeed, the cylinder geometry model shows that the full Hamiltonian may have only pure point spectrum, yet there are nontrivial edge currents. Hence, the existence of edge currents is not directly tied to the existence of continuous spectrum. We will discuss this in more detail in section 4.

3.1 The Mourre Inequality for H_0

We construct a conjugate operator for $H_0 = H_L(B) + V_0$, where the confining potential V_0 depends only on x, as above. Let $U_\alpha = \mathrm{e}^{i\alpha p_y}$, for $p_y = -i\partial_y$, and for any $\alpha \in \mathbb{R}$, be the translation group in the y-direction defined by

$$(U_\alpha g)(y) = g(y + \alpha). \tag{3.1}$$

Since the representation is unitary, the operator S_α defined by

$$S_\alpha = \frac{i}{2}(U_\alpha y - y U_{-\alpha}) \tag{3.2}$$

is easily seen to be selfadjoint on the domain D_y of the operator multiplication by y, since U_α preserves this domain.

We next compute the commutator $i[H_0, S_\alpha]$, $\alpha \in \mathbb{R}$. The operator S_α commutes with p_x and V_0. Since $V_y = p_y - Bx$, it is easy to check that

$$[V_y, S_\alpha] = \frac{1}{2}(U_\alpha - U_{-\alpha}) = i\sin(\alpha p_y), \tag{3.3}$$

so that

$$i[H_0, S_\alpha] = -2\sin(\alpha p_y)V_y, \tag{3.4}$$

as a quadratic form on $D(H_0) \cap D_y$, or as an operator identity on the core $C_0^\infty(\mathbb{R}^2)$. We also need to compute the double commutator $[[H_0, S_\alpha], S_\alpha]$. By formula (3.4), we find that

$$[[H_0, S_\alpha], S_\alpha] = 2i[\sin(\alpha p_y), V_y] = 0. \tag{3.5}$$

Consequently, a positive commutator will imply absolutely continuous spectrum (cf. [1]) in the range of the corresponding spectral projector as in Proposition 3.1.

We first derive a general expression for $\langle \psi, [H_0, iS_\alpha]\psi \rangle$, for $\psi \in E_0(\Delta_n)L^2(\mathbb{R}^2) \cap D_y$ and $\alpha \in \mathbb{R}$. For any $\psi \in D(H_0) \cap D_y$, it follows from (3.4) that

$$\langle \psi, [H_0, iS_\alpha]\psi \rangle = -2 \int_{\mathbb{R}} \sin(\alpha k) \langle \hat{\psi}(\cdot; k), \hat{V}_y \hat{\psi}(\cdot; k) \rangle \, dk, \tag{3.6}$$

where, as above, $\hat{u}$ denotes the partial Fourier transform of u with respect to y. We assume that V_0 satisfies (2.7) and choose $|\Delta_n|/B$ small enough so Lemma 2.1 holds true. Writing $\psi \in E_0(\Delta_n)L^2(\mathbb{R}^2) \cap D_y$ as in (2.4), we find that

$$\langle \psi, [H_0, iS_\alpha]\psi \rangle = -\sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)} \sin(\alpha k) |\beta_j(k)|^2 \omega_j'(k) \, dk. \tag{3.7}$$

Here we used the Feynman-Hellmann formula and the vanishing of the cross-terms established in Lemma 2.1. The potential V_0 being an even function, this is still the case for the ω_j's (see Lemma 5.1), so (3.7) can be rewritten as

$$\langle \psi, [H_0, iS_\alpha]\psi \rangle = -\sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)_-} \sin(\alpha k) (|\beta_j(k)|^2 + |\beta_j(-k)|^2) \omega_j'(k) \, dk. \tag{3.8}$$

In order to prove a Mourre estimate, it is necessary to bound the right side of (3.8) from below by a positive constant times $\|\psi\|^2$. This obviously requires a lower bound on the derivative $\omega_j'(k)$ of the dispersion curves.

We now examine the case where V_0 is either the parabolic confining potential (1.4) or the sharp confining potential (1.2).

The Parabolic Confining Potential Case

Let Δ_n be as in (2.18):

$$\Delta_n = [(2n + a)B_g, (2n + c)B_g], \; 1 < a < c < 3.$$

When the confining potential V_0 is defined by (1.4), the dispersion curves $\omega_j(k)$, $j = 0, 1, \cdots, n$, are parabolas with equation given by (2.16). Whence

$$\omega_j^{-1}(\Delta_n)_- = (-k_j(c), -k_j(a)) \cup (k_j(a), k_j(c)),$$

with $k_j(x) = B_g^{3/2}/g\sqrt{2(n-j)+x-1}$, for $x = a, c$ and $j = 0, 1, \cdots, n$, according to (2.19)-(2.20). In view of proving the coming proposition, let us notice that

$$k_j(c) > k_j(a) > k_{j+1}(c) > k_{j+1}(a) > 0,\ j = 0, 1, \ldots, n-1. \tag{3.9}$$

Proposition 3.1 *Let $|\Delta_n|/B$ be small enough so Lemma 2.1 holds true. Then for any $0 < \alpha < \pi/k_0(c)$, we have*

$$-iE_0(\Delta_n)[H_0, S_\alpha]E_0(\Delta_n) \geq \frac{2g}{B_g^{1/2}}(a-1)^{1/2} s_{\alpha,n}(a,c) E_0(\Delta_n), \tag{3.10}$$

with the constant

$$s_{\alpha,n}(a,c) \equiv \min\left(\sin(\alpha k_n(a)), \sin(\alpha k_0(c))\right) > 0. \tag{3.11}$$

Proof.
By combining (3.8) with the explicit expression (2.16) of the $\omega_j(k)$'s and bearing in mind the derivative ω_j' is an odd function, we have

$$\begin{aligned} & \langle \psi, [H_0, iS_\alpha]\psi \rangle \\ = \; & -2\left(\frac{g}{B_g}\right)^2 \sum_{j=0}^{n} \int_{-k_j(c)}^{-k_j(a)} k \sin(\alpha k)(|\beta_j(k)|^2 + |\beta_j(-k)|^2)\ dk. \end{aligned}$$

Recalling now the definition (3.11), it is easy to check the function $k\sin(\alpha k)$ is bounded from below by $k_n(a)s_\alpha(a,c)$ in $\cup_{j=0}^{n}[-k_j(c), -k_j(a)]$. This, together with the identity

$$\sum_{j=0}^{n} \int_{-k_j(c)}^{-k_j(a)} (|\beta_j(k)|^2 + |\beta_j(-k)|^2)\ dk = \|\psi\|^2,$$

proves the positivity of the commutator. ■

The Sharp Confining Potential Case

Let Δ_n be defined by (2.13):

$$\Delta_n = [(2n+a)B, (2n+c)B],\ 1 < a < c < 3.$$

As follows from Lemmas 2.2 and 2.3 under suitable conditions on B and $\mathcal{V}_0$, each set $\omega_j^{-1}(\Delta_n)_-$, $j = 0, 1, \ldots, n$, is an interval $[-k_j^+, -k_j^-]$ with :

$$0 < \frac{BL}{3} < k_j^- < k_j^+.$$

The ω_j being written in increasing order, we have in addition $k_j^\pm < k_{j-1}^\pm$ for all $j = 1, \ldots, n$, so

$$\cup_{j=0}^n \omega_j^{-1}(\Delta_n)_- \subset [-k_0^+, -k_n^-].$$

In particular the following inequality holds true,

$$0 < \frac{BL}{3} < k_n^- < k_0^+, \tag{3.12}$$

allowing us to prove the coming statement.

Proposition 3.2 *Assume that $|\Delta_n|/B$ is sufficiently small so Lemma 2.1 holds true. Then for any $0 < \alpha < \pi/k_0^+$, we have*

$$s_{\alpha,n} \equiv \min\big(\sin(\alpha k_n^-), \sin(\alpha k_0^+)\big) > 0, \tag{3.13}$$

and

$$-iE_0(\Delta_n)[H_0, S_\alpha]E_0(\Delta_n) \geq C_n(a-1)^2(3-c)^2 B^{1/2} s_{\alpha,n} E_0(\Delta_n), \tag{3.14}$$

provided B is large enough and $\mathcal{V}_0 \geq 2(2n+c)B$. The constant $C_n > 0$ is defined in Lemma 2.2 and is independent of B and $\mathcal{V}_0$.

Proof.
In light of (3.12) it is easy to check that

$$-\sin(\alpha k) \geq s_{\alpha,n},\ k \in \omega_j^{-1}(\Delta_n)_-,\ j = 0, 1, \ldots, n,$$

so the result is an obvious consequence of (3.8) and Lemma 2.2. ■

3.2 Perturbation Theory and Spectral Stability

The benefit of a local positive commutator is its stability under perturbations. We consider two types of perturbations of H_0: 1) Perturbations periodic in the y-direction, and 2) Perturbations decaying in the y-direction. As we mention below, these conditions on the perturbations are much weaker than what is required using scattering theoretic methods. In light of the positive commutator results (3.10) and (3.14), we will treat both confining potentials (1.2) and (1.4) simultaneously, only referring to the explicit lower bound for the commutator of H_0 when needed.

Perturbations Periodic in the y-Direction

We first treat perturbations $V_1(x,y)$ satisfying $V_1(x,y+T) = V_1(x,y)$, for some $T > 0$. Due to the y-periodicity of V_1, the main property we will use in this section is the following basic identity:

$$[V_1, U_T] = 0. \tag{3.15}$$

Proposition 3.3 *If the magnetic strength B is taken large enough, there is a constant $c = c(T) > 0$ such that*

$$-iE(\Delta_n)[H_0 + V_1, S_T]E(\Delta_n) \geq cB^{1/2}E(\Delta_n)$$

provided T, $|\Delta_n|/B$ and $v_1 \equiv \|V_1\|_\infty/B$ are sufficiently small.

Proof.
Let $\Delta_n \subset \tilde{\Delta}_n$ be defined as in (2.70). We decompose $\psi \in E(\Delta_n)L^2(\mathbb{R}^2)$ as in (2.74)

$$\psi = E_0(\tilde{\Delta}_n)\psi + E_0(\tilde{\Delta}_n^c)\psi \equiv \phi + \xi,$$

and apply (3.15):

$$\begin{aligned} \langle \psi, [H, iS_T]\psi \rangle &= \langle \psi, [H_0, iS_T]\psi \rangle \\ &= \langle \phi, [H_0, iS_T]\phi \rangle + G(\phi, \xi). \end{aligned} \tag{3.16}$$

Here the perturbation term $G(\phi, \xi)$ can be expressed, using the partial Fourier Transform in the y direction, as

$$\begin{aligned} G(\phi, \xi) &= \int_{\mathbb{R}} \sin(Tk)\langle \hat{\xi}(.,k), (k - Bx)\hat{\xi}(.,k)\rangle dk \\ &\quad + 2\mathrm{Re}\left(\int_{\mathbb{R}} \sin(Tk)\langle \hat{\phi}(.,k), (k - Bx)\hat{\xi}(.,k)\rangle dk \right). \end{aligned}$$

It is obviously bounded by $2\|(p_y - Bx)\xi\|\|\psi\|$, whence

$$-\langle\psi, [H, iS_T]\psi\rangle \geq -\langle\phi, [H_0, iS_T]\phi\rangle - 2\|(p_y - Bx)\xi\|\|\psi\|,$$

according to (3.16). The main term $(-\langle\phi, [H_0, iS_T]\phi\rangle)$ is treated by Proposition 3.2. Namely, for sufficiently small T (and under suitable assumptions on B and $\mathcal{V}_0$) there is a constant $C_n(T) > 0$ independent of B and $\mathcal{V}_0$ such that

$$-\langle\phi, [H_0, iS_T]\phi\rangle \geq C_n(T)(\tilde{a} - 1)^2(3 - \tilde{c})^2 B^{1/2}\|\phi\|^2. \tag{3.17}$$

Recalling (3.17), it remains to bound $\|(p_y - Bx)\xi\|$ and $\|\xi\|$ in a convenient way. We shall use the two following estimates :

$$\|\xi\| \leq \frac{c - a + 2v_1}{\tilde{c} - \tilde{a}}\|\psi\|, \tag{3.18}$$

and

$$\|(p_y - Bx)\xi\| \leq \left(\frac{(c - a + 2v_1)(2n + c + v_1)}{\tilde{c} - \tilde{a}}\right)^{1/2} B^{1/2}\|\psi\|, \tag{3.19}$$

whose proofs are postponed to the end of the demonstration.

Indeed, by combining inequalities (3.17)-(3.17) with (3.18)-(3.19), we obtain

$$\begin{aligned} -\langle\psi, [H, iS_T]\psi\rangle \quad \geq \quad & \left[C_n(T)(\tilde{a} - 1)^2(3 - \tilde{c})^2\left(1 - \left(\frac{c - a + 2v_1}{\tilde{c} - \tilde{a}}\right)^2\right)\right. \\ & \left. -2(2n + c + v_1)^{1/2}\left(\frac{c - a + 2v_1}{\tilde{c} - \tilde{a}}\right)^{1/2}\right] B^{1/2}\|\psi\|^2. \end{aligned}$$

It is clear now the prefactor of $B^{1/2}$ in the righthand side of (3.20) can be made positive by taking $c-a$ and v_1 sufficiently small relative to the difference $\tilde{c} - \tilde{a}$.

We turn now to proving (3.18)-(3.19). First, E denoting the midpoint of $\Delta_n \subset \tilde{\Delta}_n$, we notice that $(H_0 - E)^{-1}\xi$ is well defined, so we have

$$\begin{aligned} \|\xi\|^2 &= \langle\psi, \xi\rangle \\ &= \langle(H_0 - E)\psi, (H_0 - E)^{-1}\xi\rangle \\ &\leq \|(H - E - V_1)\psi\|\|(H_0 - E)^{-1}\xi\|, \end{aligned}$$

from the Cauchy-Schwarz inequality. This, together with the two following basic estimates,

$$\|(H-E-V_1)\psi\| \leq \left(\frac{|\Delta_n|}{2} + \|V_1\|_\infty\right)\|\psi\|, \tag{3.20}$$

and

$$\|(H_0-E)^{-1}\xi\| \leq \operatorname{dist}^{-1}(E, \tilde{\Delta}_n^c)\|\xi\|,$$

proves (3.18). To show (3.19), we combine the obvious inequality

$$\|(p_y - Bx)\xi\|^2 \leq \langle H_0\xi, \xi\rangle,$$

with the following identity

$$\langle H_0\xi, \xi\rangle = \langle H_0\psi, \xi\rangle = \langle (H-V_1)\psi, \xi\rangle,$$

then we use the Cauchy-Schwarz inequality once more, getting:

$$\|(p_y - Bx)\xi\|^2 \leq \|H-V_1\|)\|\psi\|\|\xi\|.$$

Now (3.19) follows from this, (3.18) and (3.20). ■

Perturbations Decaying in the y-Direction

We now consider an impurity potential $V_1 = V_1(x,y) \in L^\infty(\mathbb{R}^2)$ having "good" decay properties in the y-direction. More precisely, we assume that V_1 decays fast enough in the y-direction so $yV_1(x,y)$ remains bounded in $\mathbb{R}^2$:

$$\|yV_1\|_\infty < \infty. \tag{3.21}$$

The reason for this additional assumption is the identity,

$$\begin{aligned} & 2[V_1, iS_\alpha] \\ = \; & (V_1(x, y+\alpha) - V_1(y))U_\alpha y - (V_1(x, y-\alpha) - V_1(x,y))yU_{-\alpha}, \end{aligned}$$

obtained by a straightforward computation. This entails

$$|\langle\psi, [V_1, iS_\alpha]\psi\rangle| \leq (2\|yV_1\|_\infty + |\alpha|\|V_1\|_\infty)\|\psi\|^2, \; \psi \in D(H_0) \cap D_y, \; \alpha \in \mathbb{R},$$

which, combined with the proof of Proposition 3.3, entails:

Proposition 3.4 *Let B be large. Then there is a constant $c = c(\alpha) > 0$ such that*

$$-iE(\Delta_n)[H_0 + V_1, S_\alpha]E(\Delta_n) \geq cB^{1/2}E(\Delta_n),$$

provided α, $|\Delta_n|/B$, $\|V_1\|_\infty/B$ and $\|yV_1\|_\infty/B^{1/2}$ are sufficiently small.

Remark on the Stability of the Absolutely Continuous Spectrum for Strips

Following the idea developed by Macris, Martin and Pulé in [9] for the half-plane geometry, we can actually prove $H_0 + V_1$ has purely absolutely continuous spectrum for the two-edge geometry if the perturbation V_1 is bounded and integrable in $\mathbb{R}^2$. This class of perturbations is weaker than the classes considered above for which we proved the existence of absolutely continuous spectrum away from the Landau levels since, roughly speaking, the L^1-condition requires decay in all directions. The proof of this result relies on the diamagnetic inequality (see [1], [11]):

$$|\mathrm{e}^{-tH_L(B)}u| \leq \mathrm{e}^{t\Delta}|u|,\ u \in L^2(\mathbb{R}),\ t \in \mathbb{R}_+. \tag{3.22}$$

Here $(-\Delta)$ denotes the nonnegative Laplacian in $\mathbb{R}^2$ and (3.22) holds true for all B. As the confining potential V_0 is nonnegative in $\mathbb{R}^2$, Kato's inequality (3.22) still holds by substituting H_0 for $H_L(B)$, giving

$$|\mathrm{e}^{-tH_0}u| \leq \mathrm{e}^{t\Delta}|u| \text{ and } |\mathrm{e}^{-tH}u| \leq \mathrm{e}^{t\|V_1\|_\infty}\mathrm{e}^{t\Delta}|u|,\ u \in L^2(\mathbb{R}),\ t \in \mathbb{R}_+, \tag{3.23}$$

since V_1 is bounded. It follows by explicit calculation that $|V_1|^{1/2}\mathrm{e}^{t\Delta}$ belongs to the Schmidt class $\mathcal{B}_2(L^2(\mathbb{R}^2))$ so that the same is true for $|V_1|^{1/2}\mathrm{e}^{-tH_0}$ and $|V_1|^{1/2}\mathrm{e}^{-tH}$ by (3.23), with the following estimates:

$$\||V_1|^{1/2}\mathrm{e}^{-tH_0}\|_{\mathcal{B}_2(L^2(\mathbb{R}^2))} = \frac{\|V_1\|_1}{\sqrt{2\pi t}} \text{ and } \||V_1|^{1/2}\mathrm{e}^{-tH}\|_{\mathcal{B}_2(L^2(\mathbb{R}^2))} = \mathrm{e}^{t\|V_1\|_\infty}\frac{\|V_1\|_1}{\sqrt{2\pi t}}. \tag{3.24}$$

Let $\mathcal{B}_1(L^2(\mathbb{R}^2))$ denote the trace class. To estimate the trace norm of $\mathrm{e}^{-tH} - \mathrm{e}^{-tH_0}$, we use Duhamel's formula

$$\mathrm{e}^{-tH} = \mathrm{e}^{-tH_0} - \int_0^t \mathrm{e}^{sH}V_1\mathrm{e}^{-sH_0}ds. \tag{3.25}$$

Due to the estimates (3.24), the Hölder inequality for the trace norm, and (3.25), we obtain

$$\begin{aligned} \|\mathrm{e}^{-tH} - \mathrm{e}^{-tH_0}\|_{\mathcal{B}_1(L^2(\mathbb{R}^2))} &\leq \int_0^t \|\mathrm{e}^{(s-t)H}V_1\mathrm{e}^{-sH_0}\|_{\mathcal{B}_1(L^2(\mathbb{R}^2))} \\ &\leq \frac{\|V_1\|_1^2\mathrm{e}^{t\|V_1\|_\infty}}{2\pi}\int_0^t \frac{ds}{\sqrt{s(t-s)}} < \infty. \end{aligned} \tag{3.26}$$

Whence $\mathrm{e}^{-tH} - \mathrm{e}^{-tH_0}$ is a trace class operator for all $t > 0$ so H has an absolutely continuous spectrum by the Kato-Rosenblum Theorem and the fact that H_0 has purely absolutely continuous spectrum.

4 Bounded, Two-Edge, Cylindrical Geometry

We address now the case of a quantum device with bounded cylindrical geometry. More precisely, the charged particle is assumed to be moving on the cylinder C_D of circumference $D > 0$ and confined along the cylinder axis by two boundaries separated by the distance $L > 0$. We define the infinite cylinder as $C_D = \mathbb{R} \times J = \{(x, y) \mid x \in \mathbb{R}, y \in J\}$, where J is an interval with length D,

$$J = [-D/2, D/2],$$

and identify $y = -D/2$ with $y = D/2$. The trajectories of the particle will be bounded in the x-direction by confining potentials.

Let us give now a precise statement of the model. The Landau Hamiltonian $H_L(B) = p_x^2 + (p_y - Bx)^2$ is endowed with y-periodic boundary conditions

$$\varphi(x, -D/2) = \varphi(x, D/2) \text{ and } \partial_y \varphi(x, -D/2) = \partial_y \varphi(x, D/2), \qquad (4.1)$$

making it selfadjoint in $L^2(C_D)$. As in the preceding sections, the quantum particle is confined in the x-direction to the strip $[-L/2, L/2]$ by adding to $H_L(B)$ a confining potential $V_0 = V_0(x)$ fulfilling the condition (2.25),

$$V_0(x)\chi_{\{|x|<L/2\}}(x) = 0,$$

and condition (2.7). The spectrum of $H_0 = H_L(B) + V_0$ consists of eigenvalues for energies below C, where C is the limit at infinity of the confining potential as in (2.7). It follows from Lemma 4.1 that the entire spectrum of H_0 is discrete in the case of the power function confining potential (1.3). Despite this, we shall prove that suitable states $\varphi = E_0(\Delta_n)\varphi$, $\Delta_n \subset (E_n(B), E_{n+1}(B))$, carry a current of size $B^{1/2}$, and that this current survives in presence of a sufficiently small perturbation. Thus, the existence of the edge current is independent of the spectral type of the operator.

This result is in accordance with (and complements) the one obtained by Ferrari and Macris, who have extensively investigated this model ([4], [5], [6], [7]) in the particular case where $D = L$. They consider an Anderson-type random potential V_ω and prove with large probability (under a rather technical assumption on the spectra of the Hamiltonians $H_0^{(l)}$ and $H_0^{(r)}$ obtained respectively by removing the left or the right wall from H_0) that the spectrum of the random Hamiltonian $H_\omega = H_0 + V_\omega$ in an energy interval

$(B + \|V_\omega\|_\infty, 3B - \|V_\omega\|_\infty)$ consists in the union of two sets σ_l and σ_r. The eigenvalues in σ_α, $\alpha = l, r$, are actually small perturbations of eigenvalues $E_j^{(l)}$ of the half-plane Hamiltonian $H_0^{(\alpha)} + V_\omega$ and they show the edge current carried by an associated eigenstate $\varphi_j^{(\alpha)}$ is of size D (with opposite signs depending on whether $\alpha = l$ or r). Their analysis extends to the case where L is at least of size $\log D$.

The remaining of this section is organized as follows. After arguing $\sigma(H_0)$ is pure point, we estimate the current carried by an eigenstate of H_0 in the case of the sharp confining potential (1.2). Then we extend this estimate to the case of a convenient wave packet $\varphi = E_0(\Delta_n)\varphi$ for $\Delta_n \subset (E_n(B), E_{n+1}(B))$ and in presence of a perturbation V sufficiently small relative to B. We point out that the estimates on the edge currents given in the remaining of this section are obtained unconditionally on the size of L and B and they hold for general wave packets with energy in between two consecutive Landau levels.

4.1 Nature of the Spectrum of $H_L(B)$ and H_0

The spectrum of $H_L(B)$

Let us define the Fourier transform $\mathcal{F}$ as $\mathcal{F}\varphi(x) = (\hat{\varphi}_p(x))_{p\in\mathbb{Z}}$, where

$$\hat{\varphi}_p(x) = \int_J \varphi(x, y)\frac{\mathrm{e}^{-ik_p y}}{\sqrt{D}}dy \text{ and } k_p = \frac{2\pi}{D}p, \tag{4.2}$$

for any $p \in \mathbb{Z}$ and a.e. $x \in \mathbb{R}$. It is unitary from $L^2(C_D)$ endowed with the usual scalar product onto $l^2(\mathbb{Z}; L^2(\mathbb{R}))$. Due to the periodic boundary conditions (4.1), it is standard result that

$$\mathcal{F}H_L(B)\mathcal{F}^* = \sum_{p\in\mathbb{Z}}^{\oplus} h_L(k_p), \tag{4.3}$$

where $h_L(k)$, $k \in \mathbb{R}$, still denotes the operator $p_x^2 + (k - Bx)^2$ in $L^2(\mathbb{R})$. The spectrum of $h_L(k)$ is discrete and does not depend on k, $\sigma(h_L(k)) = (2\mathbb{N} + 1)B$, each eigenvalue being simple. For any $m \geq 0$, the normalized eigenvector $\psi_m(.; k)$ of $h_L(k)$ associated to the eigenvalue $(2m+1)B$ is given by (2.46). By setting

$$\Psi_m^{(p)}(x, y) \equiv \psi_m(x; k_p)\frac{\mathrm{e}^{ik_p y}}{\sqrt{D}}, \ m \in \mathbb{N}, \ p \in \mathbb{Z}, \tag{4.4}$$

we see from (4.3) the set $\{\Psi_m^{(p)},\ m \in \mathbb{N},\ p \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(C_D)$ which diagonalizes $H_L(B)$, in the sense that

$$H_L(B) = \sum_{m \geq 0} (2m+1)B \left(\sum_{p \in \mathbb{Z}} |\Psi_m^{(p)}\rangle\langle\Psi_m^{(p)}| \right). \tag{4.5}$$

The spectrum of $H_L(B)$ is also purely punctual with $\sigma(H_L) = (2\mathbb{N}+1)B$, each eigenvalue having infinite multiplicity.

We turn now to describing the spectrum of $H_0 = H_L(B) + V_0$.

Spectrum of H_0

The confining potential V_0 being a function of x alone we deduce from (4.3) that

$$\mathcal{F} H_0 \mathcal{F}^* = \sum_{p \in \mathbb{Z}}^{\oplus} h_0(k_p), \tag{4.6}$$

where $h_0(k)$ is still defined by (1.6). Moreover the effective potential $V_{eff}(x;k) = (Bx-k)^2 + V_0(x)$ is unbounded at infinity so the resolvent of $h_0(k)$ is compact. We recall the eigenvalues of $h_0(k)$ are denoted $\omega_m(k)$, $m \in \mathbb{N}$, the corresponding normalized eigenfunction being called $\varphi_m(x;k)$. By setting analogously to (4.4)

$$\Phi_m^{(p)}(x,y) \equiv \varphi_m(x;k_p) \frac{\mathrm{e}^{ik_p y}}{\sqrt{D}},\ m \in \mathbb{N},\ p \in \mathbb{Z}, \tag{4.7}$$

we obtain in the same way as before that $\{\Phi_m^{(p)},\ m \in \mathbb{N},\ p \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(C_D)$, and deduce from (4.6) that

$$H_0 = \sum_{m \geq 0} \sum_{p \in \mathbb{Z}} \omega_m(k_p) |\Phi_m^{(p)}\rangle\langle\Phi_m^{(p)}|. \tag{4.8}$$

This means that H_0 has pure point spectrum:

$$\sigma(H_0) = \{\omega_m(k_p),\ m \geq 0,\ p \in \mathbb{Z}\}. \tag{4.9}$$

Let us now consider an impurity potential $V_1 \in L^\infty(C_D)$ such that:

$$V_1(x,y)\chi_{\{|x|>L/2\}}(x) = 0. \tag{4.10}$$

The bounded potential V_1 also has a compact support, hence it is compact. Now, one question arising from (4.9) is to determine whether the perturbed Hamiltonian $H = H_0 + V_1$ has an eigenvalue. Since H is obtained from H_0 by adding a compact perturbation V_1, standard arguments warrant the answer is positive provided $\sigma(H_0)$ is discrete. We state in the coming lemma that this is the case for suitable unbounded confining potentials V_0.

Lemma 4.1 *The spectrum $\sigma(H_0)$ remains discrete provided V_0 is nonnegative and is such that*

$$\lim_{|x| \to +\infty} V_0(x) = +\infty. \tag{4.11}$$

Proof.
Taking account of (4.9), we need to show that each eigenvalue $E = \omega_{m_0}(k_{p_0})$, $(m_0, p_0) \in \mathbb{N} \times \mathbb{Z}$, is isolated and has finite multiplicity.

The potential V_0 being nonnegative we first notice ω_m is bounded from below by $E_m(B)$, so the set

$$M_E(\epsilon) \equiv \{m \in \mathbb{N}, \omega_m^{-1}((E - \epsilon, E + \epsilon)) \neq \emptyset\},$$

is finite for any $\epsilon > 0$. Next, V_0 satisfying (4.11) we know from Lemma 5.2 in Appendix 1 that

$$\lim_{|k| \to +\infty} \omega_m(k) = +\infty,$$

so $\omega_m^{-1}((E - \epsilon, E + \epsilon))$ is bounded for any $m \in M_E(\epsilon)$.

This indicates that $\{m \in \mathbb{N},\ p \in \mathbb{Z},\ \omega_m(k_p) \in (E - \epsilon, E + \epsilon)\}$ is necessarily a finite set, proving the result. ■

Notice that the power function confining potential (1.3) fulfills (4.11), so $\sigma_e(H) = \sigma_e(H_0) = \emptyset$ by Lemma 4.1, whence $\sigma(H)$ remains discrete in this case.

Moreover, in the particular case where V_0 is the sharp confining potential (1.2), we can argue in the same way as in the proof of Lemma 4.1 that each eigenvalue $\omega_{m_0}(k_{p_0})$ has finite multiplicity. However it is not clear that the spectrum of H_0 remains discrete. Indeed as $|p|$ goes to infinity, each $\omega_m(k_p)$, $m \in \mathbb{N}$, goes to $E_m(B) + \mathcal{V}_0$ by Lemma 5.2, so the eigenvalues lying in a neighborhood of $E_m(B) + \mathcal{V}_0$ may not be isolated.

4.2 Edge Currents: the Unperturbed Case

Let Δ_n for $n \geq 0$, be defined by (2.13),

$$\Delta_n = ((2n+a)B, (2n+c)B),\ 1 < a < c < 3,$$

and consider a state $\varphi = E_0(\Delta_n)\varphi$.

We want to estimate the current carried by φ along the edges of the free sample C_D (i.e. when the impurity potential $V_1 = 0$). It turns out (see below the estimate (4.21) of the current carried by a wave packet) this current is the weighted sum of the currents carried by all the eigenstates $\Phi_m^{(p)}$, $(m, p) \in \mathbb{N} \times \mathbb{Z}$, such that

$$\omega_m(k_p) \in \Delta_n. \tag{4.12}$$

We therefore start by estimating the current carried by such an eigenstate $\Phi_m^{(p)}$, for appropriate indices $m \in \mathbb{N}$ and $p \in \mathbb{Z}_-$. In a second step we extend this estimate to the case of the wave packet φ.

For simplicity, we assume in the remaining of this section that V_0 is the sharp confining potential (1.2).

Current Carried by an Eigenstate

We consider an eigenfunction $\Phi_m^{(p)}$ of H_0 for some (m, p) in $\mathbb{N} \times \mathbb{Z}_-$ satisfying (4.12). The current carried by $\Phi_m^{(p)}$ along the left edge of the cylinder C_D is defined as the expectation $\langle \Phi_m^{(p)}, V_y \Phi_m^{(p)} \rangle$ of the velocity operator $V_y = p_x - Bx$ in the y-direction. By recalling the formal equality $V_y = \frac{1}{2B}(V_0' + i[H_0, p_x])$, the current immediately decomposes in two terms :

$$\langle \Phi_m^{(p)}, V_y \Phi_m^{(p)} \rangle = \frac{1}{2B} \left(\langle \Phi_m^{(p)}, V_0' \Phi_m^{(p)} \rangle + \langle \Phi_m^{(p)}, [H_0, p_x] \Phi_m^{(p)} \rangle \right).$$

Following the notations of section 2.1, the second term

$$\frac{1}{2B} \langle \Phi_m^{(p)}, [H_0, p_x] \Phi_m^{(p)} \rangle = \frac{1}{2B} \langle \Phi_m^{(p)}, [H_0 - \omega_m(k_p), p_x] \Phi_m^{(p)} \rangle,$$

vanishes according to the Virial theorem, so we have

$$\langle \Phi_m^{(p)}, V_y \Phi_m^{(p)} \rangle = \omega_m'(k_p), \tag{4.13}$$

by the Feynman-Hellmann Formula. In light of (4.13) the following result follows immediately from Lemma 2.2.

Proposition 4.1 *Let V_0 be the sharp confining potential (1.2) and Δ_n be defined by (2.13), $|\Delta_n|/B$ being sufficiently small so Lemma 2.1 is true. Then, for any $(m,p) \in \mathbb{N} \times \mathbb{Z}_-$ satisfying (4.12), there is a constant $C_n > 0$ independent of B and $\mathcal{V}_0$ such that*

$$-\langle \Phi_m^{(p)}, V_y \Phi_m^{(p)} \rangle \geq C_n B^{1/2},$$

provided B is large enough.

Current Carried by a Wave Packet

We turn now to estimating the current carried along C_D by a the state $\varphi = E_0(\Delta_n)\varphi$. The state φ decomposes in the orthonormal basis $\{\Phi_m^{(p)},\ m \in \mathbb{N},\ p \in \mathbb{Z}\}$ as

$$\varphi(x,y) = \sum_{p \in \mathbb{Z}} \sum_{\substack{0 \leq m \leq n \\ \omega_m(k_p) \in \Delta_n}} \beta_m^{(p)} \Phi_m^{(p)}(x,y), \tag{4.14}$$

where

$$\beta_m^{(p)} = \langle \varphi, \Phi_m^{(p)} \rangle. \tag{4.15}$$

We suppose that $\mathcal{V}_0$ is sufficiently large, more precisely that

$$\mathcal{V}_0 \geq E_n(B) + B, \tag{4.16}$$

so there are only a finite number of index p's involved in the sum (4.14). Indeed, we know from Lemma 5.2 that $\lim_{|k| \to +\infty} \omega_0(k) = E_0(B) + \mathcal{V}_0$ with $E_0(B) + \mathcal{V}_0 \geq E_{n+1}(B)$ according to (4.16). Whence there is necessarily $p_n^* \in \mathbb{N}$ such that

$$\omega_0(k_{p_n^*}) \in \Delta_n \text{ and } \omega_0(k_p) \notin \Delta_n \text{ for all } |p| > p_n^*. \tag{4.17}$$

Since $\omega_n(k) > \omega_0(k)$ for all $n \geq 1$ and $k \in \mathbb{R}$, we see that $\omega_n(k_p) \notin \Delta_n$ for any $|p| > p_n^*$, so (4.14) finally reduces to:

$$\varphi(x,y) = \sum_{|p| \leq p_n^*} \sum_{\substack{0 \leq m \leq n \\ \omega_m(k_p) \in \Delta_n}} \beta_m^{(p)} \Phi_m^{(p)}(x,y). \tag{4.18}$$

Henceforth, the current carried by φ along the left edge of the cylinder has the following expression:

$$\langle \varphi, V_y \varphi \rangle = \sum_{|p|,|p'| \leq p_n^*} \sum_{\substack{0 \leq m, m' \leq n \\ \omega_m(k_p) \in \Delta_n \\ \omega_{m'}(k_{p'}) \in \Delta_n}} \beta_m^{(p)} \overline{\beta_{m'}^{(p')}} \langle \Phi_m^{(p)}, v_y \Phi_{m'}^{(p')} \rangle. \tag{4.19}$$

Actually the crossed terms $\langle \Phi_m^{(p)}, V_y \Phi_{m'}^{(p')} \rangle$ in (4.19) vanish for $p \neq p'$. This can be seen from the two following basic identities

$$\begin{aligned} \mathcal{F}\Phi_m^{(p)}(x) &= (\delta(s-p)\varphi_m(x;k_p))_{s\in\mathbb{Z}}\,, \\ \mathcal{F}\left(V_y \Phi_{m'}^{(p')}\right)(x) &= (\delta(s-p')(x)(k_{p'}-Bx)(x)\varphi_{m'}(x;k_{p'}))_{s\in\mathbb{Z}}\,, \end{aligned}$$

and from the unitarity of $\mathcal{F}$:

$$\begin{aligned} \langle \Phi_m^{(p)}, V_y \Phi_{m'}^{(p')} \rangle &= \delta(p'-p)\langle \varphi_m(.;k_p), (k_p - Bx)\varphi_{m'}(.;k_p)\rangle \\ &= \delta(p'-p)\langle \Phi_m^{(p)}, V_y \Phi_{m'}^{(p)} \rangle. \end{aligned}$$

As a consequence, (4.19) can be rewritten as

$$\langle \varphi, V_y \varphi \rangle = \sum_{|p|\leq p_n^*} \sum_{\substack{0\,\leq\, m,\, m'\,\leq\, n \\ \omega_m(k_p)\,\in\,\Delta_n \\ \omega_{m'}(k_p)\,\in\,\Delta_n}} \beta_m^{(p)} \overline{\beta_{m'}^{(p)}} \langle \Phi_m^{(p)}, V_y \Phi_{m'}^{(p)} \rangle. \tag{4.20}$$

Moreover, taking $|\Delta_n|/B$ sufficiently small, we have $\omega_m^{-1}(\Delta_n) \cap \omega_{m'}^{-1}(\Delta_n) = \emptyset$ for all $m \neq m'$ according to Lemma 2.1, so end up getting:

$$\langle \varphi, V_y \varphi \rangle_{L^2(C_D)} = \sum_{|p|\leq p_n^*} \sum_{\substack{0\,\leq\, m\,\leq\, n \\ \omega_m(k_p)\,\in\,\Delta_n}} |\beta_m^{(p)}|^2 \langle \Phi_m^{(p)}, V_y \Phi_m^{(p)} \rangle_{L^2(C_D)}. \tag{4.21}$$

This shows the current carried by φ is the $|\beta_m^{(p)}|^2$-weighted sum of the current carried by the eigenstates $\Phi_m^{(p)}$ with energy $\omega_m(k_p)$ in Δ_n. In light of (4.21) and Proposition 4.1 we have obtained the following result:

Proposition 4.2 *Let Δ_n be defined by (2.13) and V_0 denote the sharp confining potential (1.2). Let $\varphi \in L^2(C_D)$ satisfy $E_0(\Delta_n)\varphi = \varphi$ and p_n^* be the smallest integer satisfying (4.17), so φ has expansion as in (4.18). Assume that φ is mostly supported on the set of negative wave numbers k_p, i.e. that there is a constant $\gamma > 0$ such that the coefficients $\beta_m^{(p)}$ defined by (4.15) satisfy*

$$|\beta_m^{(-p)}|^2 \geq (1+\gamma^2)|\beta_m^{(p)}|^2, \tag{4.22}$$

for all $m = 0, 1, \ldots, n$, $p = 0, 1, \ldots, p_n^*$ such that $\omega_m(k_p) \in \Delta_n$. Then there is a constant $C_n > 0$ independent of B and $\mathcal{V}_0$ such that

$$-\langle \varphi, V_y \varphi \rangle \geq C_n \frac{\gamma^2}{2+\gamma^2}\,(a-1)^2\,(3-c)^2\,B^{1/2}\|\varphi\|^2,$$

provided B is large enough and $|c-a|$ is sufficiently small.

Proof.
For any $-p_n^* \leq p \leq 0$ and $0 \leq m \leq n$ such that $\omega_m(k_p) \in \Delta_n$, Proposition 4.1 assures us that

$$-\langle \Phi_m^{(p)}, V_y \Phi_m^{(p)} \rangle \geq C_n \,(a-1)^2 \,(3-c)^2 \,B^{1/2},$$

so the result follows from (4.21) and (4.22) by just mimicking the proof of Theorem 2.1. ■

4.3 Perturbation Theory

As in section 2.7 for the strip geometries we now consider the perturbation of the edge currents by adding a bounded impurity potential $V_1(x,y)$ to H_0, and show the lower bound on the edge currents is stable with respect to these perturbations provided $\|V_1\|_\infty$ is not too large compared with B.

We continue to use the same notation as in section 2.7. That is, $\Delta_n \subset \mathbb{R}$ denotes a closed, bounded interval with $\Delta_n \subset (E_n(B), E_{n+1}(B))$, for some $n \geq 0$. We write the interval Δ_n as in (2.13):

$$\Delta_n = [(2n+a)B, (2n+c)B], \quad \text{for } 1 < a < c < 3.$$

We consider the larger interval $\tilde{\Delta}_n$ defined by (2.70), containing Δ_n, and with the same midpoint $E \equiv (2n + (a+c)/2)B$,

$$\tilde{\Delta}_n = [(2n+\tilde{a})B, (2n+\tilde{c})B], \quad \text{for } 1 < \tilde{a} < a < c < \tilde{c} < 3.$$

By recalling Proposition 4.2 and arguing in the same way as in the proof of Theorem 2.2 we obtain the following result.

Theorem 4.1 *Let $V_0(x)$ be the sharp confining potential (1.2). Let $V_1(x,y)$ be a bounded potential and let $E(\Delta_n)$ denote the spectral projection for $H = H_0 + V_1$ and the interval Δ_n. Let $\psi \in L^2(C_D)$ be a state satisfying $\psi = E(\Delta_n)\psi$. Let $\phi \equiv E_0(\tilde{\Delta}_n)\psi$ and $\xi \equiv E_0(\tilde{\Delta}_n^c)\psi$, so that $\psi = \phi + \xi$. Let ϕ have an expansion as in (4.18) with coefficients $\beta_m^{(p)}$ satisfying the condition (4.22) of Proposition 4.2, that is:*

$$\exists \gamma > 0, \; |\beta_m^{(-p)}|^2 \geq (1+\gamma^2)|\beta_m^{(p)}|^2,$$

for all $m = 0, 1, \ldots, n$ and $p = 0, 1, \ldots, p_n^*$ such that $\omega_m(k_p) \in \Delta_n$.

Then, we have,

$$-\langle\psi, V_y\psi\rangle \geq B^{1/2}\left(\frac{\gamma^2}{2+\gamma^2}C_n(3-\tilde{c})^2(\tilde{a}-1)^2 - F_n(\|V_1\|/B)\right)\ \|\psi\|^2,$$

where $F_n(\|V_1\|/B)$ has the same expression as in (2.72). If we suppose that $\|V_1\|_\infty < v_0 B$, then for a fixed level n, if $c-a$ and v_0 are sufficiently small (depending on $\tilde{a}$, $\tilde{c}$, and n), there is a constant $D_n > 0$ so that for all B, we have

$$-\langle\psi, V_y\psi\rangle \geq D_n B^{1/2}\|\psi\|.$$

5 Appendix 1 : Basic Properties of the Eigenvalues and Eigenfunctions

Let $V_0 = V_0(x) \in L^2_{loc}(\mathbb{R})$ be nonnegative. Then the resolvent of the operator $h_0(k) = h_L(k) + V_0$ is compact since the effective potential $V_{eff}(x;k) = (Bx-k)^2 + V_0(x)$ is unbounded as $|x| \to \infty$, so the spectrum is discrete with only ∞ as an accumulation point. We write the eigenvalues of $h_0(k)$ in increasing order and denote them by $\omega_j(k)$, $j \geq 0$. The normalized eigenfunction associated to $\omega_j(k)$ is $\varphi_j(x;k)$. We recall from Proposition 7.2 in [8] that the eigenvalues $\omega_j(k)$, $j \geq 0$, are simple for all $k \in \mathbb{R}$.

In this Appendix we collect the main properties of the eigenvalues and eigenfunctions of the operator $h_0(k)$ for an even confining potential V_0.

5.1 Symmetry Properties

Lemma 5.1 *Let $V_0(x) \in L^2_{loc}(\mathbb{R})$ be a even confining potential. Then for any $j \in \mathbb{N}$ and $k \in \mathbb{R}$, the eigenvalues $\omega_j(k)$ and eigenfunctions $\varphi_j(x;k)$ of $h_0(k)$ satisfy:*

$$\begin{array}{ll} (i) & \omega_j(-k) = \omega_j(k) \\ (ii) & \varphi_j(-x;-k) = \pm\varphi_j(x;k). \end{array}$$

Proof.
The operation P that implements $x \to (-x)$ satisfies $P\text{dom}\, h_0(k) = \text{dom}\, h_0(-k)$ and $Ph_0(k) = h_0(-k)P$. This entails

$$h_0(-k)P\varphi_j(x;k) = \omega_j(k)P\varphi_j(x;k), \tag{5.1}$$

so $\omega_j(k)$ is an eigenvalue of $h_0(-k)$, and there is necessarily some $m_k \geq 0$ such that $\omega_j(k) = \omega_{m_k}(-k)$.
Since this is true for any $q \neq k$, we can find $m_q \geq 0$ such that $\omega_j(q) = \omega_{m_q}(-q)$. Moreover ω_j being a continuous function, $\omega_{m_q}(-q)$ goes to $\omega_{m_k}(-k)$ as q goes to k, so $m_q = m_k$ by the simplicity of the eigenvalues. Therefore m_k does not depend on k. By writing now m instead of m_k we have shown that

$$\omega_j(k) = \omega_m(-k),\ \forall k \in \mathbb{R}.$$

It follows in particular from this that $\omega_n(0) = \omega_m(0)$ so we immediately get $m = n$ from the simplicity of the eigenvalues once more.
To prove (ii), we substitute $(-k)$ for k in (5.1) and use (i), getting

$$h_0(k)\varphi_j(-x;-k) = \omega_j(k) P \varphi_j(-x;-k).$$

Now the result follows from the simplicity of the real valued eigenfunction $\varphi_j(x;k)$ together with the normalization condition $\|\varphi_j(.;\pm k)\| = 1$. ■

5.2 Asymptotic Behavior and Separation of the Dispersion Curves

We show below that the asymptotic behavior w.r.t. k of the eigenvalue $\omega_j(k)$, $j \in \mathbb{N}$, depends on whether the confining potential V_0 is bounded at infinity or not. More precisely, we assume V_0 satisfies (2.7): There is a generalized constant $0 < C \leq \infty$ such that

$$\begin{cases} (a) & 0 \leq V_0(x) \leq C,\ \forall x \in \mathbb{R} \\ (b) & \lim_{|x|\to\infty} V_0(x) = C. \end{cases}$$

We now deduce from the assumption (2.7) the eigenvalue $\omega_j(k)$ converges to $E_j(B) + C$ or $+\infty$, depending on whether the constant C in (2.7) is finite or infinite. As a corollary, we show in Lemma 5.3 the dispersion curves remain separated.

Asymptotic behavior of ω_j

Lemma 5.2 *Let V_0 fulfill (2.7). Then, for any $j \in \mathbb{N}$, we have:*

$$(i) \quad \lim_{|k|\to+\infty} \omega_j(k) = E_j(B) + C \quad \text{if } 0 < C < \infty \tag{5.2}$$

$$(ii) \quad \lim_{|k|\to+\infty} \omega_j(k) = +\infty \quad \text{if } C = \infty. \tag{5.3}$$

Proof.
Due to Lemma 5.1 it is enough to show the result for positive k.
Case (i). We first deduce from operator inequality $h_0(k) \leq h_L(k)+C$, which obviously follows from (2.7)(a), that

$$\omega_j(k) \leq E_j(B) + C. \tag{5.4}$$

We next fix $\varepsilon \in (0,1)$ and derive from (2.7)(b) there is necessarily $x_\varepsilon > 0$ such that

$$V_0(x) \geq C - \varepsilon,\ \forall |x| > x_\varepsilon. \tag{5.5}$$

Let φ be a normalized function in the domain of $h_0(k)$. By combining the following basic inequality

$$\langle h_L(k)\varphi, \varphi\rangle_{L^2(\mathbb{R})} \geq (1-\varepsilon)\langle h_L(k)\varphi, \varphi\rangle_{L^2(\mathbb{R})} + \varepsilon \int_{|x|\leq x_\varepsilon} (Bx-k)^2 |\varphi(x)|^2 dx,$$

with (5.5), we have

$$\begin{aligned} \langle h_0(k)\varphi, \varphi\rangle_{L^2(\mathbb{R})} &= \langle h_L(k)\varphi, \varphi\rangle_{L^2(\mathbb{R})} + \int_{\mathbb{R}} V_0(x)|\varphi(x)|^2 dx \\ &\geq (1-\varepsilon)\langle h_L(k)\varphi, \varphi\rangle_{L^2(\mathbb{R})} + C - \varepsilon + R_\varepsilon, \end{aligned} \tag{5.6}$$

where the remaining term is $R_\varepsilon \equiv \int_{|x|\leq x_\varepsilon} (\varepsilon(Bx-k)^2 - C)\,|\varphi(x)|^2 dx$. Since $\varepsilon(Bx-k)^2 - C \geq 0$ on $[-x_\varepsilon, x_\varepsilon]$ for all $k \geq k_\varepsilon \equiv Bx_\varepsilon + \sqrt{C/\varepsilon}$, (5.6) immediately leads to

$$\langle h_0(k)\varphi, \varphi\rangle_{L^2(\mathbb{R})} \geq (1-\varepsilon)\langle h_L(k)\varphi, \varphi\rangle_{L^2(\mathbb{R})} + C - \varepsilon,\ k \geq k_\varepsilon.$$

Let $\mathcal{M}_j$ denote a j-dimensional submanifold of dom $h_0(k)$, $j = 0, 1, 2, \cdots, n$. It follows from the above inequality and the Max-Min Principle that

$$\omega_j(k) \geq \min_{\varphi\in\mathcal{M}_j^\perp,\ \|\varphi\|=1} \langle h_0(k)\varphi, \varphi\rangle_{L^2(\mathbb{R})} \geq \min_{\varphi\in\mathcal{M}_j^\perp,\ \|\varphi\|=1} (1-\varepsilon)\langle h_L(k)\varphi, \varphi\rangle_{L^2(\mathbb{R})} + C - \varepsilon,$$

so we obtain

$$\omega_j(k) \geq (1-\varepsilon)E_j(B) + C - \varepsilon,\ \forall k \geq k_\varepsilon,$$

by taking the max over the $\mathcal{M}_j$'s. Now (5.2) follows from this and (5.4).
Case (ii). The function V_0 being nonnegative according to (2.7)(a), the effective potential $V_{eff}(x;k) = V_0(x) + (Bx-k)^2$ satisfies

$$V_{eff}(x;k) \geq \tilde{V}(x;k),\ x \in \mathbb{R},\ k \geq 0,$$

where

$$\tilde{V}(x;k) = \begin{cases} k^2/4 & \text{if } x \in (-\infty, k/(2B)) \cup (3k/(2B), +\infty) \\ \inf_{x\in[k/(2B),3k/(2B)]} V_0(x) & \text{if } x \in [k/(2B), 3k/(2B)]. \end{cases}$$

Hence $V_{eff}(.;k)$ is uniformly bounded from below by

$$\tilde{V}(k) \equiv \min(k^2/4, \inf_{x\in[k/(2B),3k/(2B)]} V_0(x)),$$

with $\lim_{k\to+\infty} \tilde{V}(k) = +\infty$ according to (2.7)(b). This, together with the obvious estimate $h_0(k) \geq \tilde{V}(x;k)$ proves (5.3). ∎

Separation of the Dispersion Curves

Lemma 5.3 *If V_0 satisfies (2.7), then for all $j \in \mathbb{N}$ we have*

$$(i) \quad \inf_{k\in\mathbb{R}} (\omega_{j+1}(k) - \omega_j(k)) > 0 \quad \text{if } 0 < C < +\infty. \quad (5.7)$$

$$(ii) \quad \forall X > 0,\ \inf_{|k|\leq X} (\omega_{j+1}(k) - \omega_j(k)) > 0 \quad \text{if } C = +\infty. \quad (5.8)$$

Proof.
The constant C being finite, let us suppose that

$$\inf_{k\in\mathbb{R}} (\omega_{j+1}(k) - \omega_j(k)) = 0,$$

for some $j \in \mathbb{N}$. There would also be a sequence $(k_m)_{m\geq 1}$ of real numbers, such that

$$0 \leq \omega_{j+1}(k_m) - \omega_j(k_m) < \frac{1}{m},\ m \geq 1. \quad (5.9)$$

Due to the evenness of ω_j and ω_{j+1}, the k_m could actually be chosen nonnegative, and for all $B > 0$, we deduce from Lemma 5.2 the sequence $(k_m)_{m\geq 1}$ would be necessarily bounded. Therefore we could build a subsequence

$(k_{m'})_{m'}$ of $(k_m)_m$ that converges to $k^* \in \mathbb{R}_+$. Hence, by substituting m' for m in (5.9) and taking the limit as m' goes to infinity, we would have

$$\omega_j(k^*) = \omega_{j+1}(k^*),$$

since ω_j and ω_{j+1} are continuous functions. This would mean $\omega_j(k^*)$ is a doubly-degenerated eigenvalue of $h_0(k^*)$, a contradiction to the simplicity of the eigenvalues of $h_0(k)$, $k \in \mathbb{R}$.

Evidently the case $C = +\infty$ is obtained by arguing in the same way as before since the parameters k considered in this case are taken in a bounded set. ∎

6 Appendix 2 : Technical Estimates for the Power Function Confining Potential

We collect in Lemmas 6.1 and 6.3 two technical estimates used in section 2.6 for the calculation of the lower bound (2.26) on the edge current, in the particular case where the confining potential is the power function (1.3).

Though Lemma 6.1 is actually valid for more general confining potentials, we assume for simplicity in this appendix that V_0 denotes the power function confining potential (1.3).

6.1 Bounding Eigenfunctions in the Classically Forbidden Region

Lemma 6.1 *Upon taking B sufficiently large, we have*

$$0 \leq \int_{\mathbb{R}_+} V_0(x) \varphi_j(x;k) \psi_m(x;k) dx \leq \frac{L}{2} \sqrt{(2n+c)B},$$

for all $k \in \omega_j^{-1}(\Delta_n)_-$ and $j,\ m = 0, 1, \ldots, n$.

Proof.
Let k be in $\omega_j^{-1}(\Delta_n)_-$. We know from Lemma 2.3 that $k \leq -BL/3$ provided B is sufficiently large, so the effective potential $W_j(x;k)$ defined in (2.30) is

positive in the region $x \geq 0$. As a consequence the non-zero $H^1(\mathbb{R})$-solution $\varphi_j(.;k)$ to the differential equation

$$\varphi''(x) = W_j(x;k)\varphi(x), \tag{6.1}$$

does not vanish in $\mathbb{R}_+$, according to Proposition 8.1 in [8]. Moreover $\varphi_j(L/2;k)$ being chosen positive, we have in addition:

$$\varphi_j'(x;k) < 0 \text{ and } \varphi_j''(x;k) > 0,\ x \geq 0. \tag{6.2}$$

This, together with inequality $\|\varphi_j'(.;k)\|^2 \leq \omega_j(k)$, which immediately follows from (6.1), involves

$$\varphi_j'(L/2;k)^2 \leq \frac{2}{L}\int_0^{L/2} \varphi_j'(x;k)^2 dx \leq \frac{2}{L}(2n+c)B. \tag{6.3}$$

Similarly, B being taken sufficiently large so the quadratic potential $Q_m(x;k)$ defined in (2.62) is positive in the region $x \geq 0$, the function $\psi_m(x;k)$ may be taken positive in $\mathbb{R}_+$, with

$$\psi_m'(x;k) < 0,\ x \geq 0, \tag{6.4}$$

since this is a non-zero $H^1(\mathbb{R})$-solution to the differential equation $\psi''(x) = Q_m(x;k)\psi(x)$. Using the normalization condition $\|\psi_m(.;k)\| = 1$, it follows from this that

$$\psi_m(L/2;k)^2 \leq \frac{2}{L}\int_0^{L/2} \psi_m(x;k)^2 dx \leq \frac{2}{L}. \tag{6.5}$$

We turn now to estimating the integral $\int_{\mathbb{R}_+} V_0(x)\varphi_j(x;k)\psi_m(x;k)dx$. By reference to equation (6.1) we substitute the expression $-\varphi_j''(x;k) - ((Bx-k)^2 - \omega_j(k))\varphi_j(x;k)$ for $V_0(x)\varphi_j(x;k)$ in the integrand, getting,

$$\int_0^{+\infty} V_0(x)\varphi_j(x;k)\psi_m(x;k) \leq \int_{L/2}^{+\infty} \varphi_j''(x;k)\psi_m(x;k)dx, \tag{6.6}$$

since $((Bx-k)^2 - \omega_j(k))\varphi_j(x;k)\psi_m(x;k)$ is nonnegative in the region $x \geq 0$. An integration by parts in the r.h.s. of (6.6) now provides

$$\begin{aligned}
&\int_0^{+\infty} V_0(x)\varphi_j(x;k)\psi_m(x;k) \\
\leq\ & \left|\varphi_j'(L/2;k)\right| \psi_m(L/2;k) - \int_{L/2}^{+\infty} \varphi_j'(x;k)\psi_m'(x;k)dx,
\end{aligned}$$

the last integral being positive according to (6.2) and (6.4). The result follows from this together with (6.3) and (6.5). ■

6.2 Bounding Eigenfunctions Outside the Classically Forbidden Region

Bounding the integral $\int_{(-\infty,-L/2)}(-x-L/2)^{p+1}\psi_m(x;k)^2dx$ as in Lemma 6.3 requires a slightly different strategy from the one used in the proof of Lemma 6.1. Indeed, for $k \in \omega_j^{-1}(\Delta_n)_-$, it is not guaranteed the set $(-\infty, L/2)$ is entirely in the classically forbidden region of $h_L(k)$ for the energy $(2m+1)B$. This can be seen from the fact the quadratic potential $Q_m(x;k)$ defined in (2.62) vanishes at the coordinates $x_\pm = k/B \pm \sqrt{2m+1}/B^{1/2}$, which, in light of Lemma 2.3, may belong to $(-\infty, L/2)$.

In view of Lemma 6.3 (in the particular case where V_0 is the power function confining potential (1.3)) we actually need a more precise bound from below on the set $\omega_j^{-1}(\Delta_n)_-$, than the one given by Lemma 2.3. This is the purpose of the coming Lemma.

Wave Numbers Estimate Revisited

Lemma 6.2 *Any given $\delta > 0$ we have*

$$\omega_j^{-1}(\Delta_n)_- \subset [-B(L/2+\delta) - \sqrt{(2n+c)B}, 0],\ j = 0, 1, \cdots, n,$$

provided

$$\mathcal{V}_0 \geq (2n+c)B/\delta^p. \tag{6.7}$$

Proof.
The estimation on the upper bound of $\omega_j^{-1}(\Delta_n)_-$ following immediately from its definition, it only remains to prove the estimate on the lower bound. Actually the eigenvalues $\omega_j(k)$, $j = 0, 1, \cdots, n$, of $h_0(k)$, being written in ascending order, it is enough to prove the result for $j = 0$. To do that, we consider a normalized function φ in the domain of $h_0(k)$, $k \in \mathbb{R}$, and apply

the definition (1.3) of V_0. We obtain:

$$\begin{aligned}
& \langle h_0(k)\varphi,\varphi\rangle \\
= & \ \langle h_L(k)\varphi,\varphi\rangle + \mathcal{V}_0 \int_{L/2}^{+\infty} (x-L/2)^p(|\varphi(x)|^2+|\varphi(-x)|^2)dx \\
\geq & \ \langle h_L(k)\varphi,\varphi\rangle + \mathcal{V}_0 \int_{L/2+\delta}^{+\infty} (x-L/2)^p(|\varphi(x)|^2+|\varphi(-x)|^2)dx \\
\geq & \ \langle h_L(k)\varphi,\varphi\rangle + \mathcal{V}_0\delta^p \int_{|x|\geq L/2+\delta} |\varphi(x)|^2dx, \qquad (6.8)
\end{aligned}$$

for any $\delta > 0$. Using the normalization condition $\|\varphi\| = 1$ together with the obvious operators comparison $h_L(k) \geq (Bx-k)^2$, we deduce from (6.8) that

$$\begin{aligned}
& \langle h_0(k)\varphi,\varphi\rangle \\
\geq & \ \mathcal{V}_0\delta^p + \int_{|x|<L/2+\delta} ((Bx-k)^2 - \mathcal{V}_0\delta^p)|\varphi(x)|^2dx. \qquad (6.9)
\end{aligned}$$

Let us assume now that $k \leq k_\delta \equiv -B\,(L/2+\delta) - \sqrt{\mathcal{V}_0\delta^p}$ so $\omega_0(k) \geq \mathcal{V}_0\delta^p$ for all $k \leq k_\delta$ from the Min-Max Principle. This means that

$$\inf \omega_0^{-1}(\Delta_n)_- \geq -B\,(L/2+\delta) + \sqrt{(2n+c)B}, \qquad (6.10)$$

in the particular case where $\mathcal{V}_0 = (2n+c)B/\delta^p$. To achieve the proof it is enough to notice that (6.10) remains valid for $\mathcal{V}_0 > (2n+c)B/\delta^p$ since $\omega_0(k)$ is an increasing function of $\mathcal{V}_0$. ■

Armed with this Lemma we can prove the main result of this section.

The Main Result

Lemma 6.3 *There is a constant $C_m(n,p) > 0$ independent of B and $\mathcal{V}_0$ such that,*

$$\int_{-\infty}^{-L/2} (-x-L/2)^{p+1}\psi_m(x;k)^2dx \leq C_m(n,p)B^{-\frac{p+1}{2}}, \ k \in \omega_j^{-1}(\Delta_n)_-, \qquad (6.11)$$

provided

$$\mathcal{V}_0 \geq (2n+c)B^{\frac{p+2}{2}}. \qquad (6.12)$$

Proof.
Let us define the constant

$$h_m \equiv \sup_{u\in\mathbb{R}} H_m(u)\mathrm{e}^{-u^2/4},$$

where H_m still denotes the m^{th} Hermite polynomial function. The main ingredient of the proof is the following estimate

$$|\psi_m(x;k)| \leq \left(\frac{B}{\pi}\right)^{1/4} \frac{h_m}{\sqrt{2^m m!}}\mathrm{e}^{-B/4(x-k/B)^2},\ k\in\omega_j^{-1}(\Delta_n)_-, \tag{6.13}$$

which obviously follows from the explicit expression (2.46) of $\psi_m(x;k)$. Indeed, by substituting the r.h.s. of (6.13) for $\psi_m(x;k)$ in the integral in (6.11), we obtain

$$\begin{aligned} &\int_{-\infty}^{-L/2} (-x-L/2)^{p+1}\psi_m(x;k)^2 dx \\ \leq\ & \left(\frac{B}{\pi}\right)^{1/2} \frac{h_m^2}{2^m m!}\int_{L/2}^{+\infty} (x-L/2)^{p+1}\mathrm{e}^{-B/2(x+k/B)^2}dx, \end{aligned} \tag{6.14}$$

so we are left with the task of bounding the preceding integral, called J_p in the remaining of this proof. To do that, we study two cases separately.
First Case: $k/B \geq -L/2$. In this case, it is enough to notice that $x+k/B \geq x-L/2 \geq 0$ for all $x \geq L/2$, and use the change of variable $t=\sqrt{B}(x-L/2)$, getting

$$\begin{aligned} J_p &\leq \int_{L/2}^{+\infty} (x-L/2)^{p+1}\mathrm{e}^{-B/2(x-L/2)^2}dx \\ &\leq \left(\int_0^{+\infty} t^{p+1}\mathrm{e}^{-t^2/2}dt\right) B^{-\frac{p+2}{2}}, \end{aligned} \tag{6.15}$$

so (6.11) immediately follows from this and from (6.14).
Second Case: $k/B < -L/2$. Let us decompose the integral J_p into two terms :

$$\begin{aligned} J_p &= \int_{L/2}^{-(L/2+2k/B)} (x-L/2)^{p+1}\mathrm{e}^{-B/2(x+k/B)^2}dx \\ &+ \int_{-(L/2+2k/B)}^{+\infty} (x-L/2)^{p+1}\mathrm{e}^{-B/2(x+k/B)^2}dx. \end{aligned} \tag{6.16}$$

The first integral can be treated by applying Lemma 6.2 for $\delta = B^{-1/2}$. We get that

$$0 \leq x - L/2 \leq -2(L/2 + k/B) \leq 2B^{-1/2}(1 + \sqrt{2n+c}), \tag{6.17}$$

provided (6.12) is satisfied, this last condition being obtained by simply rewriting (6.7) with $\delta = B^{-1/2}$. This, together with the change of variable $t = \sqrt{B}(x + k/B)$ involves:

$$\begin{aligned} & \int_{L/2}^{-(L/2+2k/B)} (x - L/2)^{p+1} \mathrm{e}^{-B/2(x+k/B)^2} dx \\ \leq \quad & 2^{p+1}(1 + \sqrt{2n+c})^{p+1} \sqrt{\pi} B^{-(p+2)/2}. \end{aligned} \tag{6.18}$$

The bound on the second term in (6.16) is obtained by noticing that

$$0 < x - L/2 \leq 2(x + k/B), \text{ for all } x \geq -(L/2 + 2k/B),$$

and using the change of variable $t = \sqrt{B}(x + k/B)$ once more:

$$\begin{aligned} & \int_{-(L/2+2k/B)}^{+\infty} (x - L/2)^{p+1} \mathrm{e}^{-B/2(x+k/B)^2} dx \\ \leq \quad & \left(2^{p+1} \int_0^{+\infty} t^{p+1} \mathrm{e}^{-t^2/2} dt \right) B^{-(p+2)/2}. \end{aligned}$$

In light of (6.15), the result now follows from this, (6.14), (6.16) and (6.18). ■

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